Harmonic analysis related to homogeneous varieties in three dimensional vector spaces over finite fields

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Abstract. In this paper we study the extension problem, the averaging problem, and the generalized Erdős-Falconer distance problem associated with arbitrary homogeneous varieties in three dimensional vector spaces over finite fields. In the case when the varieties do not contain any plane passing through the origin, we obtain the best possible results on aforementioned three problems. In particular, our result on the extension problem modestly generalizes the result by Mockenhaupt and Tao who studied the particular conical extension problem. In addition, investigating the Fourier decay on homogeneous varieties enables us to give complete mapping properties of averaging operators. Moreover, we improve the size condition on a set such that the cardinality of its distance set is nontrivial.

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1. Introduction

Both the extension problem and the averaging problem ask us to determine the boundedness of certain operators. In the Euclidean space, these problems have been well studied, but have not yet been solved in any higher dimensions or general settings. On the other hand, the Falconer distance problem is considered as a continuous analogue of the Erdős distance problem, which roughly asks that how large a distance set can be. The recent breakthrough work [8] by Guth and Katz has confirmed the Erdős distance conjecture in dimension two, which says the distance set is essentially as large as the set. However the Falconer distance problem remains open in any dimension, and the current best results were obtained by Erdoğan [5] who applied Tao's bilinear restriction theorem [27], one of the most beautiful theorems in harmonic analysis. Motivated by these work, these problems have been recently studied in the finite field setting in part finite fields not only serve as a typical model for the Euclidean space but also have powerful structures which enable us to relate

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the above problems to other well-studied problems in arithmetic combinatorics, algebraic geometry, and analytic number theory. Moreover, finite fields often yield better results compared to the same problems in the Euclidean setting. In this paper, we shall focus on studying the following well-known Euclidean problems related to homogeneous varieties in $\mathbb{F}_q^3$: the extension problem, the averaging problem, and the Erdős-Falconer distance problem.

Before we introduce our main results, let us briefly review these problems in the Euclidean setting. Let H be a set in $\mathbb{R}^d$ and $d\sigma$ a measure on the set H. In the Euclidean case, the extension problem is to determine the optimal range of exponents $1 \leq p, r \leq \infty$ such that the following extension estimate holds:

$$\|(fd\sigma)^\vee\|_{L^r(\mathbb{R}^d)} \leq C(p,r,d)\|f\|_{L^p(H,d\sigma)} \text{ for all } f \in L^p(H,d\sigma)$$

where $(fd\sigma)^\vee$ denotes the inverse Fourier transform of the measure $fd\sigma$. This problem was first addressed in 1967 by Stein [22] and it has been extensively studied in the last few decades. We refer readers to [28] for a comprehensive survey of this problem.

The averaging problem also asks us to find the exponents $1 \leq p, r \leq \infty$ such that the following inequality holds:

$$\|f * d\sigma\|_{L^r(\mathbb{R}^d)} \leq C(p,r,d)\|f\|_{L^p(\mathbb{R}^d)} \text{ for all } f \in L^p(\mathbb{R}^d), \tag{1.1}$$

where $d\sigma$ is a measure supported on a surface H in $\mathbb{R}^d$ and the convolution $f * d\sigma$ is defined by the relation $f * d\sigma(x) = \int_H f(x-y)\, d\sigma(y)$ for $x \in \mathbb{R}^d$. For classical results on this problem, see [23], [21], and [17]. In particular, Iosevich and Sawyer [12] obtained the sharp mapping properties of averaging operators on a graph of homogeneous function of degree ≥ 2.

The Erdős distance problem and the Falconer distance problem are problems to measure sizes of distance sets determined by discrete sets and continuous sets respectively. Given $E, G \subset \mathbb{R}^d, d \geq 2$, the distance set $\Delta(E,G)$ is defined by

$$\Delta(E,G) = \{|x-y| : x \in E, y \in G\},$$

where $|\cdot|$ denotes the usual Euclidean norm. Given finite sets E, G, the Erdős distance problem is to determine the smallest possible cardinality of the distance set $\Delta(E,G)$ in terms of sizes of sets E, G. In the case when $E = G$, Erdős [6] first studied this problem and conjectured that for every finite set $E \subset \mathbb{R}^d$,

$$|\Delta(E,E)| \gtrapprox |E|^{\frac{2}{d}},$$

where $|\cdot|$ denotes the cardinality of the finite set (see [24], [25], and [26] for the best results in dimensions $d \geq 3$). As a continuous analog of the Erdős distance problem, Falconer [7] conjectured that if the Hausdorff dimension of a Borel subset E in $\mathbb{R}^d, d \geq 2$, is greater than $d/2$, then the Lebesgue measure of the distance set $\Delta(E,E)$ must be positive. This problem is known as the Falconer distance problem and is still open in all dimensions. The best known result on this problem is due to Erdoğan [5] who extended the work [30] by Wolff showing that any Borel set E with the Hausdorff dimension greater than $d/2+1/3$ yields the distance set $\Delta(E,E)$ with a positive Lebesgue measure.

The purpose of this paper is to obtain the sharp results on the extension problem, the averaging problem, and the Erdős Falconer distance problem associated with arbitrary homogeneous varieties in three dimensional vector spaces over finite fields. In the finite fields setting, we shall prove that all these problems can be completely understood by observing that an extremely good Fourier decay estimate is valid on any non-degenerate homogeneous variety in $\mathbb{F}_q^3$, which may be a specific property in the finite field setting. This fact is interesting because it could not be true in higher dimensions $d \geq 4$. For instance, if $d \geq 4$ is even, then there exists a non-degenerate homogeneous variety containing a large $d/2$-dimensional subspace. As a result, the Fourier decay estimate may not be enough to obtain the sharp averaging estimates and one would need a new approach (e.g. see [14]). On the other hand, if $d \geq 5$ is odd, then a homogeneous variety can not contain any $d/2$-dimensional subspace, because $d/2$ is not an integer. In this case, a relatively small $(d-1)/2$-dimensional subspace could be only contained in the homogeneous variety and the Fourier decay estimate could be so good that one may obtain the complete mapping properties of averaging operators. However, computing the Fourier decay estimate on homogeneous varieties in higher odd dimensions ($d \geq 5$, d is odd) is extremely difficult. Moreover, even if one succeeds to obtain good Fourier decay estimates and obtain a sharp $L^2 - L^r$ extension estimate, it is known from [14] that the result does not cover the necessary conditions for the problem in higher odd dimensions $d \geq 5$. Therefore, it is very interesting to remark that one can settle down both the extension problem and the averaging problem in three dimensions by simply using the Fourier decay estimate.

2. Notation, definitions, and key lemmas

Let $\mathbb{F}_q$ be a finite field with q elements. We denote by $\mathbb{F}_q^d, d \geq 2$, the d-dimensional vector space over the finite field $\mathbb{F}_q$. Given a set $E \subset \mathbb{F}_q^d$, we denote by $|E|$ the cardinality of the set E. For nonnegative real numbers A, B, we write $A \lesssim B$ if $A \leq CB$ for some $C > 0$ independent of the size of the underlying finite field $\mathbb{F}_q$. In other words, the constant $C > 0$ is independent of the parameter q. We also use $A \sim B$ to indicate $A \lesssim B \lesssim A$. We say that a polynomial $P(x) \in \mathbb{F}_q[x_1, \ldots, x_d]$ is a homogeneous polynomial of degree k if the polynomial's monomials with nonzero coefficients all have the same total degree k. For example, $P(x_1, x_2, x_3) = x_1^5 + x_2^3 x_3^2$ is a homogeneous polynomial of degree five. Given a homogeneous polynomial $P(x) \in \mathbb{F}_q[x_1, \ldots, x_d]$, we define a homogeneous variety H in $\mathbb{F}_q^d$ by the set

$$H = \{x \in \mathbb{F}_q^d : P(x) = 0\}.$$

For example, the cone in three dimension, which was introduced in [19], is a homogeneous variety generated by the homogeneous polynomial $P(x) = x_1^2 - x_2 x_3$. We now review the Fourier transform of a function defined on $\mathbb{F}_q^d$. Denote by χ the nontrivial additive character of $\mathbb{F}_q$. For example, if q is prime, then we may take $\chi(t) = e^{2\pi i t/q}$ where we identify $t \in \mathbb{F}_q$ with a usual integer. We now endow the space $\mathbb{F}_q^d$ with a normalized counting measure dx. Thus, given a complex valued function $f : \mathbb{F}_q^d \to \mathbb{C}$, the Fourier transform of f is defined by

$$\widehat{f}(m) = \int_{\mathbb{F}_q^d} \chi(-m \cdot x) f(x)\, dx = \frac{1}{q^d} \sum_{x \in \mathbb{F}_q^d} \chi(-m \cdot x) f(x),$$

where m is any element in the dual space of $(\mathbb{F}_q^d, dx)$. Recall that the Fourier transform $\widehat{f}$ is actually defined on the dual space of $(\mathbb{F}_q^d, dx)$. We shall endow the dual space of $(\mathbb{F}_q^d, dx)$ with a counting measure dm. We write $(\mathbb{F}_q^d, dm)$ for the dual space of $(\mathbb{F}_q^d, dx)$. Then, we also see that the Fourier inversion theorem says that for every $x \in (\mathbb{F}_q^d, dx)$,

$$f(x) = \int_{\mathbb{F}_q^d} \chi(x \cdot m) \widehat{f}(m)\, dm = \sum_{m \in \mathbb{F}_q^d} \chi(x \cdot m) \widehat{f}(m). \tag{2.1}$$

We also recall the Plancherel theorem: $\|\widehat{f}\|_{L^2(\mathbb{F}_q^d, dm)} = \|f\|_{L^2(\mathbb{F}_q^d, dx)}$, which is same as

$$\sum_{m \in \mathbb{F}_q^d} |\widehat{f}(m)|^2 = \frac{1}{q^d} \sum_{x \in \mathbb{F}_q^d} |f(x)|^2.$$

For instance, if f is a characteristic function on the subset E of $\mathbb{F}_q^d$, then the Plancherel theorem yields

$$\sum_{m\in\mathbb{F}_q^d} |\widehat{E}(m)|^2 = \frac{|E|}{q^d}, \tag{2.2}$$

here, and throughout the paper, we identify the set $E \subset \mathbb{F}_q^d$ with the characteristic function on the set E, and we denotes by $|E|$ the cardinality of the set $E \subset \mathbb{F}_q^d$.

Remark 2.1. We use the notation $\mathbb{F}_q^d$ for both the space and its dual space for a simple notation. However, this may make readers confused, because the measures of the space and its dual space are different. To overcome this confusion, we always use the variable "x" as an element of the space $(\mathbb{F}_q^d, dx)$ with the normalized counting measure dx. For example, we write $x \in \mathbb{F}_q^d$ for $x \in (\mathbb{F}_q^d, dx)$. On the other hand, we always use the variable "m" as an element of the dual space $(\mathbb{F}_q^d, dm)$ with a counting measure dm. Thus, $m \in \mathbb{F}_q^d$ means that $m \in (\mathbb{F}_q^d, dm)$.

2.1. Fourier decay estimate on homogeneous varieties. We shall estimate the Fourier transform of characteristic functions on homogeneous varieties in three dimensional vector space over the finite field $\mathbb{F}_q$. First, let us review the well-known Schwartz-Zippel lemma, which gives us the information about the cardinality of any variety in $\mathbb{F}_q^d$. For a nice proof of the Schwartz-Zippel lemma below, see Theorem 6.13 in [18].

Lemma 2.2 (Schwartz-Zippel). *Let $P(x) \in \mathbb{F}_q[x_1, \cdots, x_d]$ be a nonzero polynomial of degree k. Then, we have*

$$|\{x \in \mathbb{F}_q^d : P(x) = 0\}| \leq kq^{d-1}.$$

Using the Schwartz-Zippel lemma, we obtain the following lemma.

Lemma 2.3. *Given a nonzero homogeneous polynomial $P(x) \in \mathbb{F}_q[x_1, x_2, x_3]$, let H be the homogeneous variety given by*

$$H = \{x \in \mathbb{F}_q^3 : P(x) = 0\}.$$

If the homogeneous variety H does not contain any plane passing through the origin, then we have for every $m \in \mathbb{F}_q^3 \setminus \{(0,0,0)\}$,

$$|H \cap \Pi_m| \lesssim q,$$

where $\Pi_m = \{x \in \mathbb{F}_q^3 : m \cdot x = 0\}$ which is a hyperplane passing through the origin.

Proof. First, let us observe the set $H \cap \Pi_m$. Fix $m \neq (0,0,0)$. Without loss of generality, we may assume that $m = (m_1, m_2, -1)$. Then, we see that

$$\Pi_m = \{x \in \mathbb{F}_q^3 : m_1x_1 + m_2x_2 - x_3 = 0\}.$$

and

$$H = \{x \in \mathbb{F}_q^3 : P(x) = 0\}.$$

Thus, we see that

$$H \cap \Pi_m = \{(x_1, x_2, m_1x_1 + m_2x_2) \in \mathbb{F}_q^3 : P(x_1, x_2, m_1x_1 + m_2x_2) = 0\}.$$

Put $R(x_1, x_2) = P(x_1, x_2, m_1x_1 + m_2x_2)$. Then, it is clear that

$$|H \cap \Pi_m| = |\{(x_1, x_2) \in \mathbb{F}_q^2 : R(x_1, x_2) = 0\}|.$$

If $R(x_1, x_2)$ is a nonzero polynomial, then the Schwartz-Zippel lemma tells us that $|H \cap \Pi_m| \lesssim q$ and we complete the proof. Now assume $R(x_1, x_2)$ is a zero polynomial. Then, it follows that $R(x_1, x_2) = P(x_1, x_2, m_1x_1 + m_2x_2) = 0$ for all $x_1, x_2 \in \mathbb{F}_q$. This implies that the variety $H = \{x \in \mathbb{F}_q^3 : P(x) = 0\}$ contains a plane $m_1x_1 + m_2x_2 - x_3 = 0$, which contradicts to our hypothesis that H does not contain any plane passing through the origin. Thus, the proof is complete. $\square$

In order to compute the Fourier transform on homogeneous varieties, we shall need the following Lemma 2.4. We remark that the proof of Lemma 2.4 below adopts the invariant property of homogeneous varieties which was already observed in [4]. For readers' convenience, we state the lemma in a slightly different way and give a proof here.

LEMMA 2.4. *Let $P(x) \in \mathbb{F}_q[x_1, \ldots, x_d]$ be a nonzero homogeneous polynomial. Define a homogeneous variety $H \subset \mathbb{F}_q^d$ by*

$$H = \{x \in \mathbb{F}_q^d : P(x) = 0\}.$$

For each $m \in \mathbb{F}_q^d$, we have

$$\widehat{H}(m) = \frac{1}{q^{d+1} - q^d} \left(q|H \cap \Pi_m| - |H| \right), \tag{2.3}$$

where $\Pi_m = \{x \in \mathbb{F}_q^d : m \cdot x = 0\}$.

PROOF. For each $m \in \mathbb{F}_q^d$, we have

$$\widehat{H}(m) = q^{-d} \sum_{x \in H} \chi(-m \cdot x).$$

Since $P(x)$ is a homogeneous polynomial, a change of the variable yields that for each $t \neq 0$,

$$\widehat{H}(m) = \widehat{H}(tm).$$

It therefore follows that

$$\widehat{H}(m) = q^{-d}(q-1)^{-1} \sum_{x \in H} \sum_{t \in \mathbb{F}_q \setminus \{0\}} \chi(-tm \cdot x)$$

$$= q^{-d}(q-1)^{-1} \sum_{x \in H} \sum_{t \in \mathbb{F}_q} \chi(-tm \cdot x) - q^{-d}(q-1)^{-1}|H|.$$

By the orthogonality relation of nontrivial additive character χ, we complete the proof. □

From lemma 2.3 and 2.4, the Fourier transform on homogeneous varieties in dimension three can be estimated. The following corollary shall make a crucial role in proving our results.

COROLLARY 2.5. *Suppose the homogeneous variety $H = \{x \in \mathbb{F}_q^3 : P(x) = 0\}$ does not contain any plane passing through the origin in $\mathbb{F}_q^3$, where $P(x)$ is a homogeneous polynomial in $\mathbb{F}_q[x_1, x_2, x_3]$. Then, for any $m \neq (0,0,0)$, we have*

$$|\widehat{H}(m)| = \left| \frac{1}{q^3} \sum_{x \in H} \chi(-m \cdot x) \right| \lesssim q^{-2}. \tag{2.4}$$

PROOF. Since the homogeneous variety H does not contain any plane passing through the origin, it is clear that the polynomial $P(x) \in \mathbb{F}_q[x_1, x_2, x_3]$ is a nonzero polynomial. Thus, the Schwartz-Zippel lemma says that $|H| \lesssim q^2$ and so Corollary 2.5 follows immediately from Lemma 2.4 and Lemma 2.3. □

REMARK 2.6. Let $d\sigma$ be the normalized surface measure on the homogeneous variety $H \subset \mathbb{F}_q^3$ given in Corollary 2.5. Then, we notice that if $|H| \sim q^2$, then the conclusion (2.4) in Corollary 2.5 implies that for every $m \in \mathbb{F}_q^3 \setminus \{(0,0,0)\}$,

$$|(d\sigma)^\vee(m)| = \left| \frac{1}{|H|} \sum_{x \in H} \chi(m \cdot x) \right| \lesssim q^{-1}. \tag{2.5}$$

3. Extension problems for finite fields

In the finite field setting, Mockenhaupt and Tao [19] first set up and studied the extension problem for various algebraic varieties. Here, we review the definition of the extension problem for finite fields and introduce our main result on the problem for homogeneous varieties in three dimension. For a fixed polynomial $P(x) \in \mathbb{F}_q[x_1, \cdots, x_d]$, consider an algebraic variety

$$V = \{x \in \mathbb{F}_q^d : P(x) = 0\}.$$

Recall from Remark 2.1 that the variety V is considered as a subset of the space $(\mathbb{F}_q^d, dx)$ with the normalized counting measure dx. Therefore, if $f : (\mathbb{F}_q^d, dx) \to \mathbb{C}$ is a complex valued function, then for $1 \leq p < \infty$ the L^p-norm of f takes the following value:

$$\|f\|_{L^p(\mathbb{F}_q^d, dx)} = \left(\frac{1}{q^d} \sum_{x \in \mathbb{F}_q^d} |f(x)|^p \right)^{\frac{1}{p}}.$$

As usual, $\|f\|_{L^\infty(\mathbb{F}_q^d, dx)}$ is the maximum value of $|f|$. We now endow the variety V with the normalized surface measure $d\sigma$ such that the total mass of V is one. In other words, the surface measure $d\sigma$ supported on V can be defined by the relation

$$d\sigma(x) = \frac{q^d}{|V|} V(x)\ dx, \tag{3.1}$$

here, recall that we identify the set $V \subset \mathbb{F}_q^d$ with the characteristic function χ_V on the set V. Thus, we see that

$$\|f\|_{L^p(V, d\sigma)} = \left(\int_V |f(x)|^p d\sigma(x) \right)^{\frac{1}{p}} = \left(\frac{1}{|V|} \sum_{x \in V} |f(x)|^p \right)^{\frac{1}{p}},$$

and the inverse Fourier transform of measure $f d\sigma$ is given by

$$(f d\sigma)^\vee(m) = \int_V \chi(m \cdot x) f(x)\ d\sigma(x) = \frac{1}{|V|} \sum_{x \in V} \chi(m \cdot x) f(x),$$

where we recall that m is an element of the dual space $(\mathbb{F}_q^d, dm)$ with the counting measure dm. In addition, note that for $1 \leq p, r < \infty$,

$$\|(f d\sigma)^\vee\|_{L^r(\mathbb{F}_q^d, dm)} = \left(\int_{\mathbb{F}_q^d} |(f d\sigma)^\vee(m)|^r dm \right)^{\frac{1}{r}} = \left(\sum_{m \in \mathbb{F}_q^d} |(f d\sigma)^\vee(m)|^r \right)^{\frac{1}{r}}$$

and $\|(f d\sigma)^\vee\|_{L^\infty(\mathbb{F}_q^d, dm)}$ takes the maximum value of $|(f d\sigma)^\vee|$.

3.1. Definition of the extension theorem. Let $1 \leq p, r \leq \infty$. We denote by $R^*(p \to r)$ to be the smallest constant such that for all functions f on V,

$$\|(f d\sigma)^\vee\|_{L^r(\mathbb{F}_q^d, dm)} \leq R^*(p \to r) \|f\|_{L^p(V, d\sigma)}.$$

By duality, we note that the quantity $R^*(p \to r)$ is also the best constant such that the following restriction estimate holds: for every function g on $(\mathbb{F}_q^d, dm)$,

$$\|\widehat{g}\|_{L^{p'}(V, d\sigma)} \leq R^*(p \to r) \|g\|_{L^{r'}(\mathbb{F}_q^d, dm)}, \tag{3.2}$$

where p' and r' denote the dual exponents of p and r respectively, which mean that $1/p + 1/p' = 1$ and $1/r + 1/r' = 1$.

Observe that $R^*(p \to r)$ is always a finite number but it may depend on the parameter q, the size of the underlying finite field $\mathbb{F}_q$. In the finite field setting, the extension problem is to determine the exponents $1 \le p, r \le \infty$ such that

$$R^*(p \to r) \le C,$$

where the constant $C > 0$ is independent of q. A direct calculation yields the trivial estimate, $R^*(1 \to \infty) \lesssim 1$. Using Hölder's inequality and the nesting properties of L^p-norms, we also see that

$$R^*(p_1 \to r) \le R^*(p_2 \to r) \quad \text{for } 1 \le p_2 \le p_1 \le \infty$$

and

$$R^*(p \to r_1) \le R^*(p \to r_2) \quad \text{for } 1 \le r_2 \le r_1 \le \infty.$$

In order to obtain the strong result on the restriction problem, if $1 \le p(\text{or } r) \le \infty$, then we only need to find the smallest number $1 \le r$ (or p) $\le \infty$ such that $R^*(p \to r) \lesssim 1$. In addition, using the interpolation theorem, it therefore suffices to find the critical exponents $1 \le p, r \le \infty$.

3.2. Necessary conditions for $R^*(p \to r) \lesssim 1$. In [19], Mockenhaupt and Tao showed that if $|V| \sim q^{d-1}$ and the variety $V \subset \mathbb{F}_q^d$ contains an α-dimensional affine subspace $\Pi(|\Pi| = q^\alpha)$, then the necessary conditions for $R^*(p \to r) \lesssim 1$ are given by

$$r \ge \frac{2d}{d-1} \quad \text{and } r \ge \frac{p(d-\alpha)}{(p-1)(d-1-\alpha)}. \tag{3.3}$$

Now, let us consider the homogeneous variety $H = \{x \in \mathbb{F}_q^3 : P(x) = 0\}$ in three dimension where $P(x) \in \mathbb{F}_q[x_1, x_2, x_3]$ is a homogeneous polynomial. In addition, assume that $|H| \sim q^2$. It is clear that the homogeneous variety H contains a line, because if $P(x_0) = 0$ for some $x_0 \ne (0,0,0)$, then $P(tx_0) = 0$ for all $t \in \mathbb{F}_q$. From (3.3), we therefore see that the necessary conditions for $R^*(p \to r) \lesssim 1$ take the following:

$$r \ge 3 \quad \text{and } r \ge \frac{2p}{p-1}.$$

In particular, if $H = \{x \in \mathbb{F}_q^3 : x_1^2 - x_2x_3 = 0\}$ which is a cone in three dimension, then above necessary conditions can be improved by the conditions:

$$r \ge 4 \quad \text{and } r \ge \frac{2p}{p-1}.$$

This was proved by Mockenhaupt and Tao (see Proposition 7.1 in [19]). Moreover, they proved that $R^*(2 \to 4) \lesssim 1$ which implies that the necessary conditions are in fact sufficient conditions. Thus, the $L^2 - L^4$ extension estimate would imply the best possible results on the extension problem related to arbitrary homogeneous varieties in three dimension. It is unknown if there exists a specific homogeneous variety in $\mathbb{F}_q^3$ which yields the better extension estimates than the conical extension estimates. However, it is easy to see that there exists a homogeneous variety on which the best possible extension estimates are worser than the conical extension estimates. For example, if we take a homogeneous variety as $H = \{x \in \mathbb{F}_q^3 : x_1 + x_2 + x_3 = 0\}$, then H contains a plane, a 2-dimensional subspace, and so the necessary conditions in (3.3) say that only trivial $L^p - L^\infty, 1 \le p \le \infty$, estimates hold. Based on this example, one may ask what kind of homogeneous varieties in $\mathbb{F}_q^3$ yields the same extension estimates as the conical extension estimates? In the following section, we shall give the answer.

4. Main result on the extension problem

We prove that if the homogeneous variety in $\mathbb{F}_q^3$ does not contain any plane passing through the origin, then the extension estimates are as good as the conical extension estimates. More precisely, we have the following main result.

THEOREM 4.1. *For each homogeneous polynomial $P(x) \in \mathbb{F}_q[x_1, x_2, x_3]$, let $H = \{x \in \mathbb{F}_q^3 : P(x) = 0\}$. Suppose that $|H| \sim q^2$ and the homogeneous variety H does not contain any plane passing through the origin. Then, we have the following extension estimate on H:*

$$R^*(2 \to 4) \lesssim 1.$$

We shall give two different proofs of Theorem 4.1. One is based on geometric approach and the other is given in view of the Fourier decay estimate on the homogeneous variety in dimension three.

4.1. The proof of Theorem 4.1 based on geometric approach. In order to show that $R^*(2 \to 4) \lesssim 1$, we shall use the following well-known lemma for reduction, which is basically to estimating the incidences between the variety and its nontrivial translations. For a complete proof of the following lemma, see both Lemma 5.1 in [19] and Lemma 13 in [10].

LEMMA 4.2. *Let V be any algebraic variety in $\mathbb{F}_q^d, d \geq 2$, with $|V| \sim q^{d-1}$. Suppose that for every $\xi \in \mathbb{F}_q^d \setminus \{(0, \dots, 0)\}$,*

$$\sum_{(x,y) \in V \times V : x+y=\xi} 1 \lesssim q^{d-2}.$$

Then, we have

$$R^*(2 \to 4) \lesssim 1.$$

Using Lemma 4.2, the following lemma shall give the complete proof of Theorem 4.1.

LEMMA 4.3. *Let $P(x) \in \mathbb{F}_q[x_1, x_2, x_3]$ be a homogeneous polynomial. Suppose that the homogeneous variety $H = \{x \in \mathbb{F}_q^3 : P(x) = 0\}$ does not contain any plane passing through the origin. Then, we have that for every $\xi \in \mathbb{F}_q^3 \setminus \{(0,0,0)\}$,*

$$|\{(x, y) \in H \times H : x + y = \xi\}| \lesssim q.$$

PROOF. The first observation is that since H is a homogeneous variety, H is exactly the union of lines passing through the origin. To see this, just note that if $P(x) = 0$ for some $x \neq (0,0,0)$, then $P(tx) = 0$ for all $t \in \mathbb{F}_q$. Therefore, we can write

$$H = \bigcup_{j=1}^{N} L_j, \tag{4.1}$$

where N is a fixed positive integer, L_j denotes a line passing through the origin, and $L_i \cap L_j = \{(0,0,0)\}$ for $i \neq j$. From the Schwartz-Zippel lemma, it is clear that $|H| \lesssim q^2$. Thus, the number of lines, denoted by N, is $\lesssim q$, because each line contains q elements. The second important observation is that if H does not contain any plane passing through the origin, then for every $m \in \mathbb{F}_q^3 \setminus \{(0,0,0)\}$,

$$|H \cap \Pi_m| \lesssim q, \tag{4.2}$$

where $\Pi_m = \{x \in \mathbb{F}_q^3 : m \cdot x = 0\}$. This observation follows from Lemma 2.3. We are ready to prove our lemma. For each $\xi \neq (0,0,0)$, it suffices to prove that the number of common solutions of $P(x) = 0$ and $P(\xi - x) = 0$ is $\lesssim q$. Since $P(x)$ is a homogeneous polynomial, we see that $P(\xi - x) = 0$ if and only if $P(x - \xi) = 0$. Therefore, we aim to show that for every $\xi \neq (0,0,0)$,

$$|H \cap (H + \xi)| \lesssim q,$$

where $H+\xi=\{(x+\xi)\in\mathbb{F}_q^3 : x\in H\}$. Now, fix $\xi\neq(0,0,0)$. From (4.1), we see that

$$|H\cap(H+\xi)|\leq\sum_{j=1}^{N}|H\cap(L_j+\xi)|.$$

Notice that if $\xi\in L_j$, then $L_j+\xi=L_j$ and so $|H\cap(L_j+\xi)|=q$. However, there is at most one line L_j such that $\xi\in L_j$. Thus, it is enough to show that if $\xi\notin L_j$, then $|H\cap(L_j+\xi)|\lesssim 1$, because $N\lesssim q$. However, this will be clear from (4.2). To see this, first notice that if $(0,0,0)\neq\xi\notin L_j$, then the line $L_j+\xi$ does not pass through the origin, because the line L_j passes through the origin. Thus, the line $L_j+\xi$ is different from all lines L_k in $H=\cup_{k=1}^{N}L_k$, and so there is at most one intersection point of the line $L_j+\xi$ and each line in H. Next, consider the unique plane Π_m which contains the line $L_j+\xi$. Then, (4.2) implies that at most few lines in H lie in the plane Π_m containing the line $L_j+\xi$. Thus, we conclude that $|H\cap(L_j+\xi)|\lesssim 1$ for $\xi\notin L_j$. Thus, the proof is complete. $\square$

4.2. Remark on the Fourier decay estimate on homogeneous varieties. In [19], Mockenhaupt and Tao showed that if $d\sigma$ is the normalized surface measure on the paraboloid $V=\{x\in\mathbb{F}_q^d : x_d=x_1^2+\cdots+x_{d-1}^2\}, d\geq 2$, then the sharp Fourier decay estimate of $d\sigma$ is given by

$$|(d\sigma)^{\vee}(m)|\lesssim q^{-\frac{d-1}{2}}\quad\text{for } m\in\mathbb{F}_q^d\setminus\{(0,\ldots,0)\}. \tag{4.3}$$

Using the Tomas-Stein type argument for finite fields with the estimate, they observed that

$$R^*\left(2\to\frac{2d+2}{d-1}\right)\lesssim 1,$$

where the exponents $p=2, r=(2d+2)/(d-1)$ are called the standard Tomas-Stein exponents. In particular, if $d=3$, then $R^*(2\to 4)\lesssim 1$ which is exactly same as the conclusion of Theorem 4.1. Since (2.5) in Remark 2.6 says that the surface measure on our homogeneous variety in dimension three yields the good Fourier decay estimate in (4.3), it is clear that the Tomas-Stein type argument gives the complete proof of Theorem 4.1. However, this is not true any more if $d\geq 4$ is even. For example, if $S_0=\{x\in\mathbb{F}_q^d : x_1^2+\cdots+x_d^2=0\}$ and $d\geq 4$ is even, then the explicit Gauss sum estimates show that if $m_1^2+\cdots+m_d^2=0$, then

$$|(d\sigma)^{\vee}(m)|\sim q^{-\frac{d-2}{2}},$$

which is much weaker than the estimate (4.3). It seems that every non-degenerate homogeneous variety would yield the standard Tomas-Stein exponents in odd dimensions but it would not in even dimensions. In the last section, we shall have more discussions about it.

4.3. The proof of Theorem 4.1 by the Fourier decay estimate. In the previous subsection, we have seen that the Tomas-Stein type argument for finite fields will yield the alternative proof of Theorem 4.1. For the sake of completeness, we give the complete proof. Let $R^*: L^p(H,d\sigma)\to L^r(\mathbb{F}_q^3,dm)$ be the extension map $f\to(fd\sigma)^{\vee}$, and $R: L^{r'}(\mathbb{F}_q^3,dm)\to L^{p'}(H,d\sigma)$ be its dual, the restriction map $g\to\widehat{g}|_H$. Observe that $R^*Rg=(\widehat{g}d\sigma)^{\vee}=g*(d\sigma)^{\vee}$ for every function g on $(\mathbb{F}_q^3,dm)$. Now, in order to prove Theorem 4.1 we must show that for every function f on $(H,d\sigma)$,

$$\|(fd\sigma)^{\vee}\|_{L^4(\mathbb{F}_q^3,dm)}\lesssim\|f\|_{L^2(H,d\sigma)},$$

where $d\sigma$ is the normalized surface measure on the homogeneous variety $H\subset\mathbb{F}_q^3$. By duality (3.2), it is enough to show that the following restriction estimate holds: for every function g defined on $(\mathbb{F}_q^3,dm)$, we have

$$\|\widehat{g}\|^2_{L^2(H,d\sigma)}\lesssim\|g\|^2_{L^{\frac{4}{3}}(\mathbb{F}_q^3,dm)}.$$

By the orthogonality principle and Hölder's inequality , we see that

$$\|\widehat{g}\|^2_{L^2(H,d\sigma)} = < Rg,\ Rg >_{L^2(H,d\sigma)} = < R^*Rg,\ g >_{L^2(\mathbb{F}_q^3,dm)}$$
$$= < g * (d\sigma)^\vee,\ g >_{L^2(\mathbb{F}_q^3,dm)} \leq \|g * (d\sigma)^\vee\|_{L^4(\mathbb{F}_q^3,dm)}\ \|g\|_{L^{\frac{4}{3}}(\mathbb{F}_q^3,dm)}.$$

It therefore suffices to show that for every function g on $(\mathbb{F}_q^3, dm)$,

$$\|g * (d\sigma)^\vee\|_{L^4(\mathbb{F}_q^3,dm)} \lesssim \|g\|_{L^{\frac{4}{3}}(\mathbb{F}_q^3,dm)}.$$

For each $m \in (\mathbb{F}_q^3, dm)$, define $K(m) = (d\sigma)^\vee(m) - \delta_0(m)$ where $\delta_0(m) = 0$ for $m \neq (0,0,0)$ and $\delta_0(0,0,0) = 1$. Since $(d\sigma)^\vee(0,0,0) = 1$, we see that $K(m) = 0$ if $m = (0,0,0)$ and $K(m) = (d\sigma)^\vee(m)$ if $m \neq (0,0,0)$. It follows that

$$\|g * (d\sigma)^\vee\|_{L^4(\mathbb{F}_q^3,dm)} = \|g * (K + \delta_0)\|_{L^4(\mathbb{F}_q^3,dm)} \leq \|g * K\|_{L^4(\mathbb{F}_q^3,dm)} + \|g * \delta_0\|_{L^4(\mathbb{F}_q^3,dm)}.$$

Since $g * \delta_0 = g$ and dm is the counting measure, we see that

$$\begin{aligned} \|g * \delta_0\|_{L^4(\mathbb{F}_q^3,dm)} &= \|g\|_{L^4(\mathbb{F}_q^3,dm)} \\ &\leq \|g\|_{L^{\frac{4}{3}}(\mathbb{F}_q^3,dm)} \end{aligned}.$$

Thus, it is enough to show that for every g on $(\mathbb{F}_q^3, dm)$,

$$\|g * K\|_{L^4(\mathbb{F}_q^3,dm)} \lesssim \|g\|_{L^{\frac{4}{3}}(\mathbb{F}_q^3,dm)}.$$

However, this estimate follows immediately by interpolating the following two inequalities:

$$\|g * K\|_{L^2(\mathbb{F}_q^3,dm)} \lesssim q\|g\|_{L^2(\mathbb{F}_q^3,dm)} \tag{4.4}$$

and

$$\|g * K\|_{L^\infty(\mathbb{F}_q^3,dm)} \lesssim q^{-1}\|g\|_{L^1(\mathbb{F}_q^3,dm)}. \tag{4.5}$$

Thus, it remains to show that both (4.4) and (4.5) hold. Using the Plancherel theorem, the inequality (4.4) follows from the following observation:

$$\begin{aligned} \|g * K\|_{L^2(\mathbb{F}_q^3,dm)} &= \|\widehat{g}\widehat{K}\|_{L^2(\mathbb{F}_q^3,dx)} \\ &\leq \|\widehat{K}\|_{L^\infty(\mathbb{F}_q^3,dx)}\|\widehat{g}\|_{L^2(\mathbb{F}_q^3,dx)} \\ &\lesssim q\|g\|_{L^2(\mathbb{F}_q^3,dm)}, \end{aligned}$$

where the last line is based on the observation that for each $x \in (\mathbb{F}_q^3, dx)$
$\widehat{K}(x) = d\sigma(x) - \widehat{\delta_0}(x) = q^3|H|^{-1}H(x) - 1 \lesssim q$, because $|H| \sim q^2$ and δ_0 is a function on $(\mathbb{F}_q^3, dm)$ with a counting measure dm. Finally, the estimate (4.5) follows from Young's inequality and the Fourier decay estimate (2.5) in Remark 2.6. Thus, the proof is complete.

5. Averaging problem for finite fields

In the finite field setting, Carbery, Stones and Wright [2] recently addressed the averaging problem over algebraic varieties related to vector-valued polynomials. Recall that $(\mathbb{F}_q^d, dx), d \geq 2$, is the d-dimensional vector space with the normalized counting measure dx. For $1 \leq k \leq d-1$, they considered a specific vector-valued polynomial $P_k : \mathbb{F}_q^k \to \mathbb{F}_q^d$ given by

$$P_k(x) = \Big(x_1, x_2, \ldots, x_k, x_1^2 + x_2^2 + \cdots + x_k^2, x_1^3 + \cdots + x_k^3, \ldots, x_1^{d-k+1} + \cdots + x_k^{d-k+1}\Big)$$

and studied the averaging problem over the k-dimensional surface $V_k = \{P_k(x) \in \mathbb{F}_q^d : x \in \mathbb{F}_q^k\}$. Using the Weil's theorem [29] for exponential sums, they obtained the sharp, good Fourier decay estimates on the surface V_k, which led to the complete solution for the averaging problem. It will be also interesting to study the averaging problem over some algebraic varieties which can not be explicitly defined by a vector-valued polynomial. Koh [14] studied the averaging problem over the variety $V = \{x \in \mathbb{F}_q^d : a_1x_1^2 + a_2x_2^2 + \cdots + a_dx_d^2 = 0\}$ for all $a_j \neq 0$. Using the explicit Gauss sum estimates, he observed that if the dimension d is odd, then the sharp Fourier decay estimates on the variety V are given by $|\widehat{V}(m)| \lesssim q^{-(d+1)/2}$ for all $m \neq (0, \ldots, 0)$. In addition, he showed that if the dimension $d \geq 3$ is odd, then the complete solution to the averaging problem over the variety V can be obtained by simply applying the Fourier decay estimates on V. However, when the dimension $d \geq 2$ is even, it was also observed that the sharp Fourier decay estimates on V take the worse form: $|\widehat{V}(m)| \lesssim q^{-d/2}$ for every $m \neq (0, \ldots, 0)$ and so the averaging problem becomes much harder. From Koh's observations, one may guess that most homogeneous varieties in odd dimensions yield the good Fourier decay but the homogeneous varieties in even dimensions do not. Authors in this paper do not know the exact answer to this issue but Corollary 2.5 gives the positive answer to it in three dimensions. In this section, we shall show that Corollary 2.5 implies the complete solution to the averaging problem over the homogeneous varieties in three dimensions.

5.1. Definition of the averaging problem for finite fields. We review the averaging problem over algebraic varieties in the finite field setting. Let V be an algebraic variety in $(\mathbb{F}_q^d, dx)$, $d \geq 2$, where dx also denotes the normalized counting measure. We also denotes by $d\sigma$ the normalized surface measure on V. For $1 \leq p, r \leq \infty$, define $A(p \to r)$ as the smallest constant such that for every f defined on $(\mathbb{F}_q^d, dx)$, we have

$$\|f * d\sigma\|_{L^r(\mathbb{F}_q^d, dx)} \leq A(p \to r)\|f\|_{L^p(\mathbb{F}_q^d, dx)},$$

where we recall that $f * d\sigma(x) = \int_V f(x-y)d\sigma(y) = \frac{1}{|V|}\sum_{y \in V} f(x-y)$. Then, the averaging problem is to determine the exponents $1 \leq p, r \leq \infty$ such that $A(p \to r) \leq C$ for some constant $C > 0$ independent of q, the size of the underlying finite field $\mathbb{F}_q$.

5.2. Sharp boundedness of the averaging operator on homogeneous varieties in $\mathbb{F}_q^3$. Now, let us consider the homogeneous variety

$$H = \{x \in \mathbb{F}_q^3 : P(x) = 0\}, \tag{5.1}$$

where $P(x) \in \mathbb{F}_q[x_1, x_2, x_3]$ is a homogeneous polynomial. The following theorem is our main theorem whose proof is based on applying well-known harmonic analysis methods for the Euclidean case. We shall prove our main theorem by adopting the arguments in [**2**].

Theorem 5.1. *Let $H \subset \mathbb{F}_q^3$ be the homogeneous variety given as in (5.1). Assume that $|H| \sim q^2$ and H does not contain any plane passing through the origin. Then, we have that $A(p \to r) \lesssim 1$ if and only if $(1/p, 1/r)$ is contained in the convex hull of the points $(0,0), (0,1), (1,1), (3/4, 1/4)$.*

Remark 5.2. In the Euclidean case, it is well known that if $1 \leq r < p \leq \infty$, then $L^p - L^r$ estimate is impossible. However, in the finite field setting, we shall see that it is always true that $R^*(p \to r) \lesssim 1$ for $1 \leq r < p \leq \infty$. Like the Euclidean case, the main interest for finite fields will be the case when $1 \leq p \leq r \leq \infty$.

Proof. We prove Theorem 5.1.
$(\Longrightarrow)$ Suppose that $A(p \to r) \lesssim 1$ for $1 \leq p, r \leq \infty$. Then, it must be true that for every function f on $(\mathbb{F}_q^3, dx)$,

$$\|f * d\sigma\|_{L^r(\mathbb{F}_q^3, dx)} \lesssim \|f\|_{L^p(\mathbb{F}_q^3, dx)}.$$

In particular, this inequality also holds when we take $f = \delta_0$, where $\delta_0(x) = 0$ if $x \neq (0,0,0)$ and $\delta_0(0,0,0) = 1$. Thus, we see that

$$\|\delta_0 * d\sigma\|_{L^r(\mathbb{F}_q^3, dx)} \lesssim \|\delta_0\|_{L^p(\mathbb{F}_q^3, dx)}. \tag{5.2}$$

Since dx is the normalized counting measure, the right hand side is given by

$$\|\delta_0\|_{L^p(\mathbb{F}_q^3, dx)} = q^{-\frac{3}{p}}. \tag{5.3}$$

To estimate the left hand side, we recall from (3.1) that $d\sigma(x) = q^3|H|^{-1}H(x)\,dx$ and notice that

$$(\delta_0 * d\sigma)(x) = \frac{q^3}{|H|}(\delta_0 * H)(x) = \frac{1}{|H|}\delta_H(x),$$

where $\delta_H(x) = 1$ if $x \in H$, and $\delta_H(x) = 0$ if $x \notin H$. Thus, the left hand side in (5.2) is given by

$$\|\delta_0 * d\sigma\|_{L^r(\mathbb{F}_q^3, dx)} = q^{-\frac{3}{r}}|H|^{\frac{1-r}{r}} \sim q^{\frac{-2r-1}{r}}, \tag{5.4}$$

where we also used the hypothesis that $|H| \sim q^2$. Thus, from (5.2), (5.3), and (5.4), it must be true that

$$\frac{3}{p} \leq \frac{1}{r} + 2. \tag{5.5}$$

By duality we also see that it must be true that

$$\frac{3}{r'} \leq \frac{1}{p'} + 2.$$

From this and (5.5), a simple calculation shows that $(1/p, 1/r)$ must be contained in the convex hull of the points $(0,0), (0,1), (1,1), (3/4, 1/4)$.

($\Longleftarrow$) We must show that $A(p \to r) \lesssim 1$ for all $1 \leq p, r \leq \infty$ such that $(1/p, 1/r)$ lies in the convex hull of the points $(0,0), (0,1), (1,1), (3/4, 1/4)$. To do this, first we shall prove that for every function f on $(\mathbb{F}_q^3, dx)$,

$$\|f * d\sigma\|_{L^r(\mathbb{F}_q^3, dx)} \lesssim \|f\|_{L^p(\mathbb{F}_q^3, dx)} \quad \text{if } 1 \leq r \leq p \leq \infty. \tag{5.6}$$

Next, we shall prove that for every function f on $(\mathbb{F}_q^3, dx)$,

$$\|f * d\sigma\|_{L^4(\mathbb{F}_q^3, dx)} \lesssim \|f\|_{L^{\frac{4}{3}}(\mathbb{F}_q^3, dx)}. \tag{5.7}$$

Finally, interpolating (5.6) and (5.7) shall give the complete proof. Now, let us prove that (5.6) holds. Since $d\sigma$ is the normalized surface measure and dx is the normalized counting measure, we see that both $d\sigma$ and $(\mathbb{F}_q^3, dx)$ have total mass 1. It therefore follows from Young's inequality and Hölder's inequality that if $1 \leq r \leq p \leq \infty$, then

$$\|f * d\sigma\|_{L^r(\mathbb{F}_q^3, dx)} \leq \|f\|_{L^r(\mathbb{F}_q^3, dx)} \leq \|f\|_{L^p(\mathbb{F}_q^3, dx)}. \tag{5.8}$$

To complete the proof, it therefore suffices to show that the inequality (5.7) holds. As before, we consider a function K on $(\mathbb{F}_q^3, dm)$ defined as $K = (d\sigma)^\vee - \delta_0$. Note that for each $x \in (\mathbb{F}_q^3, dx)$, we have $\widehat{\delta_0}(x) = \int_{\mathbb{F}_q^3} \chi(-x \cdot m)\delta_0(m)dm = 1$, because dm is the counting measure. Since $d\sigma = \widehat{K} + \widehat{\delta_0} = \widehat{K} + 1$ and $\|f * 1\|_{L^4(\mathbb{F}_q^3, dx)} \lesssim \|f\|_{L^{\frac{4}{3}}(\mathbb{F}_q^3, dx)} \|1\|_{L^2(\mathbb{F}_q^3, dx)} = \|f\|_{L^{\frac{4}{3}}(\mathbb{F}_q^3, dx)}$ by Young's inequality, it is enough to show that for every f on $(\mathbb{F}_q^3, dx)$, we have

$$\|f * \widehat{K}\|_{L^4(\mathbb{F}_q^3, dx)} \lesssim \|f\|_{L^{\frac{4}{3}}(\mathbb{F}_q^3, dx)}.$$

However, this inequality can be obtained by interpolating the following two estimates:

$$\|f * \widehat{K}\|_{L^2(\mathbb{F}_q^3, dx)} \lesssim q^{-1}\|f\|_{L^2(\mathbb{F}_q^3, dx)} \tag{5.9}$$

and

$$\|f * \widehat{K}\|_{L^\infty(\mathbb{F}_q^3, dx)} \lesssim q\|f\|_{L^1(\mathbb{F}_q^3, dx)}. \tag{5.10}$$

Thus, it remains to prove that both (5.9) and (5.10) hold. From the definition of K and (2.5) in Remark 2.6, it is clear that

$$\|K\|_\infty \lesssim q^{-1}.$$

Thus, using this fact, the inequality (5.9) follows from the Plancherel theorem. On the other hand, the inequality (5.10) follows from Young's inequality and the observation that $\|\widehat{K}\|_{L^\infty(\mathbb{F}_q^3,dx)} \lesssim q$. Thus, we complete the proof. $\square$

6. Erdős-Falconer distance problem for finite fields

Let $E, F \subset \mathbb{F}_q^d, d \geq 2$. Given a polynomial $P(x) \in \mathbb{F}_q[x_1, \ldots, x_d]$, the generalized distance set $\Delta_P(E, F)$ can be defined by

$$\Delta_P(E,F) = \{P(x-y) \in \mathbb{F}_q : x \in E, y \in F\},$$

throughout this paper, we always assume the characteristic of $\mathbb{F}_q$ is larger than the degree of P. In the finite field case, the generalized Erdős distance problem is to determine the minimum cardinality of $\Delta_P(E, F)$ in terms of $|E|$ and $|F|$. In the case when $E = F$ and $P(x) = x_1^2 + x_2^2$, this problem was first introduced by Bourgain, Katz, and Tao [1]. Using the discrete Fourier analytic machinery, Iosevich and Rudnev [11] formulated this problem and obtained several interesting results. For example, they proved the following:

Theorem 6.1. *If $E \subset \mathbb{F}_q^d, d \geq 2$, with $|E| \geq Cq^{\frac{d}{2}}$ for $C > 0$ sufficiently large, then we have*

$$|\Delta_P(E,E)| \gtrsim \min\left(q, |E|q^{-\frac{d-1}{2}}\right),$$

where $P(x) = x_1^2 + \cdots + x_d^2$.

In addition, they addressed the Falconer distance problem for finite fields, which is the problem to determine the size of E such that $|\Delta_P(E, E)| \gtrsim q$. Note that Theorem 6.1 implies that if $P(x) = x_1^2 + \cdots + x_d^2$ and $|E| \gtrsim q^{(d+1)/2}$, then $|\Delta_P(E, E)| \gtrsim q$. Authors in [9] observed that if the dimension $d \geq 3$ is odd, then the exponent $(d + 1)/2$ gives the best possible result on the Falconer distance problem for finite fields. On the other hand, it has been conjectured that the exponent $d/2$ could be the best possible one if the dimension $d \geq 2$ is even. In the case when $d = 2$, the sharp exponent $(d + 1)/2$ for odd dimensions was improved by $4/3$ (see [3] or [15]). From these facts, one may think that improving Theorem 6.1 for even dimensions is only interesting. However, we shall focus on the problem in odd dimensions. The main point we want to address is that if the dimension $d \geq 3$ is odd, then the condition in Theorem 6.1, $|E| \geq Cq^{\frac{d}{2}}$, can be relaxed. On the other hand, the condition is necessary for even dimensions. More generally, we consider the following conjecture.

Conjecture 6.2. *Let $P(x) = \sum_{j=1}^d a_j x_j^c \in \mathbb{F}_q[x_1, \ldots, x_d]$ with $a_j \neq 0, c \geq 2$. If $E, F \subset \mathbb{F}_q^d$ and $d \geq 3$ is odd, then we have*

$$|\Delta_P(E,F)| \gtrsim \min\left(q, q^{-\frac{d-1}{2}}\sqrt{|E||F|}\right).$$

Authors in [16] proved that the conclusion in Conjecture 6.2 holds for all dimensions $d \geq 2$ if we assume that $|E||F| \geq Cq^d$ for a sufficiently large constant $C > 0$ (see Corollary 3.5 in [16]). They also introduced a simple example to show that if the dimension d is even, then the assumption $|E||F| \geq Cq^d$ is necessary. In addition, they pointed out that Conjecture 6.2 is true if $c = 2$. In this section, we shall prove that Conjecture 6.2 is true in the case when the dimension d is three. Observe that if Conjecture 6.2 is true, then the distance set has its nontrivial cardinality for $|E||F| \geq Cq^{d-1}$ with d odd.

6.1. Main result on the Erdős-Falconer distance problem.

In this subsection, we prove the following main theorem.

Theorem 6.3. *In dimension three, Conjecture 6.2 is true.*

First, we derive a formula for proving Theorem 6.3. Let $P(x) \in \mathbb{F}_q[x_1, \ldots, x_d]$ be a polynomial with degree ≥ 2. For each $t \in \mathbb{F}_q$, define a variety $H_t \subset \mathbb{F}_q^d$, by the set

$$H_t = \{x \in \mathbb{F}_q^d : P(x) = t\}.$$

Then, we have the following distance formula.

Lemma 6.4. *Suppose that for every $m \in \mathbb{F}_q^d \setminus \{(0, \ldots, 0)\}$ and $t \in \mathbb{F}_q$, we have*

$$|\widehat{H_t}(m)| \lesssim q^{-\frac{d+1}{2}}. \tag{6.1}$$

Then, if $E, F \subset \mathbb{F}_q^d$, then we have

$$|\Delta_P(E, F)| \gtrsim \min\left(q, q^{-\frac{d-1}{2}} \sqrt{|E||F|}\right).$$

Proof. First, we consider a counting function ν on $\mathbb{F}_q$, given by

$$\nu(t) = |\{(x, y) \in E \times F : P(x - y) = t\}| = |\{(x, y) \in E \times F : x - y \in H_t\}|.$$

Recall that $\Delta_P(E, F) = \{P(x - y) \in \mathbb{F}_q : x \in E, y \in F\}$ and notice that

$$|E||F| = \sum_{t \in \Delta_P(E,F)} \nu(t) \leq \left(\max_{t \in \mathbb{F}_q} \nu(t)\right) \; |\Delta_P(E, F)|. \tag{6.2}$$

Thus, the estimate for the upper bound of $\max_{t \in \mathbb{F}_q} \nu(t)$ is needed. For each $t \in \mathbb{F}_q$, applying the Fourier inversion theorem (2.1) to the function $H_t(x - y)$, and then using the definition of the Fourier transform, we see that

$$\nu(t) = \sum_{x \in E, y \in F} H_t(x - y) = q^{2d} \sum_{m \in \mathbb{F}_q^d} \overline{\widehat{E}}(m) \widehat{F}(m) \widehat{H_t}(m).$$

Now, write $\nu(t)$ by

$$\begin{aligned} \nu(t) =& q^{2d} \overline{\widehat{E}}(0, \ldots, 0) \widehat{F}(0, \ldots, 0) \widehat{H_t}(0, \ldots, 0) + q^{2d} \sum_{m \in \mathbb{F}_q^d \setminus \{(0, \ldots, 0)\}} \overline{\widehat{E}}(m) \widehat{F}(m) \widehat{H_t}(m) \\ =& \mathrm{I} + \mathrm{II}. \end{aligned}$$

From the definition of the Fourier transform and the Schwartz-Zippel lemma, it follows that

$$|\mathrm{I}| = \frac{1}{q^d} |E||F||H_t| \lesssim q^{-1} |E||F|.$$

On the other hand, our hypothesis (6.1) and the Cauchy-Schwarz inequality yield

$$|\mathrm{II}| \lesssim q^{2d} q^{-\frac{d+1}{2}} \left(\sum_m \left|\overline{\widehat{E}}(m)\right|^2\right)^{\frac{1}{2}} \left(\sum_m \left|\widehat{F}(m)\right|^2\right)^{\frac{1}{2}}.$$

Applying the Plancherel theorem (2.2), we obtain

$$|\mathrm{II}| \lesssim q^{\frac{d-1}{2}} |E|^{\frac{1}{2}} |F|^{\frac{1}{2}}.$$

Thus, it follows that

$$\max_{t \in \mathbb{F}_q} \nu(t) \lesssim q^{-1} |E||F| + q^{\frac{d-1}{2}} |E|^{\frac{1}{2}} |F|^{\frac{1}{2}}.$$

From this fact and (6.2), a direct calculation completes the proof. □

It seems that the assumption (6.1) in Lemma 6.4 is too strong. For example, if dimension d is even, $H_t = \{x \in \mathbb{F}_q^d : x_1^c + \cdots + x_d^c = t\}, c \geq 2$, and $u^c = -1$ for some $u \in \mathbb{F}_q$, then this case can not satisfy the assumption (6.1). This follows from a simple observation that if $E = F = \{(t_1, ut_1, \ldots, t_{d/2}, ut_{d/2}) \in \mathbb{F}_q^d : t_j \in \mathbb{F}_q\}$, then $|E| = |F| = q^{d/2}$ and $|\Delta_P(E,F)| = |\{0\}| = 1$, which does not satisfy the conclusion of Lemma 6.4. However, observe that if the dimension d is odd, then the similar example does not exist. For this reason, Conjecture 6.2 looks true. In fact, the following lemma says that only H_0 in the previous example violates the assumption (6.1).

Lemma 6.5 (4.4.19 in [4]). *Let $P(x) = \sum_{j=1}^{d} a_j x_j^s \in \mathbb{F}_q[x_1, \ldots, x_d]$ with $s \geq 2, a_j \neq 0$ for all $j = 1, \ldots, d$. In addition, assume that the characteristic of $\mathbb{F}_q$ is sufficiently large so that it does not divide s. Then,*

$$|\widehat{H_t}(m)| = \frac{1}{q^d} \left| \sum_{x \in H_t} \chi(-x \cdot m) \right| \lesssim q^{-\frac{d+1}{2}} \quad \text{for all } m \in \mathbb{F}_q^d \setminus \{(0, \ldots, 0)\}, t \in \mathbb{F}_q \setminus \{0\},$$

and

$$|\widehat{H_0}(m)| \lesssim q^{-\frac{d}{2}} \quad \text{for all } m \in \mathbb{F}_q^d \setminus \{(0, \ldots, 0)\}, \tag{6.3}$$

where $H_t = \{x \in \mathbb{F}_q^d : P(x) = t\}$.

6.2. Complete proof of Theorem 6.3. From Lemma 6.5 and Lemma 6.4, it suffices to prove that for every $m \neq (0,0,0)$,

$$|\widehat{H_0}(m)| \lesssim q^{-2},$$

where $H_0 = \{x \in \mathbb{F}_q^3 : a_1 x_1^c + a_2 x_2^c + a_3 x_3^c = 0\}$ with $a_j \neq 0, j = 1,2,3, c \geq 2$. From Corollary 2.5, it is enough to show that H_0 does not contain any plane passing through the origin in $\mathbb{F}_q^3$. By contradiction, assume that H_0 contains a plane $\Pi_m = \{x \in \mathbb{F}_q^3 : m \cdot x = 0\}$ for some $m \neq (0,0,0)$. Without loss of generality, assume that $\Pi_m = \{x \in \mathbb{F}_q^3 : x_3 = m_1' x_1 + m_2' x_2\}$ for some $m_1', m_2' \in \mathbb{F}_q$. Then, $|H_0 \cap \Pi_m| = |\Pi_m| = q^2$. However, this is impossible if q is sufficiently large. To see this, notice that

$$\begin{aligned} |H_0 \cap \Pi_m| &= |\{(x_1, x_2, m_1' x_1 + m_2' x_2) \in \mathbb{F}_q^3 : a_1 x_1^c + a_2 x_2^c + a_3 (m_1' x_1 + m_2' x_2)^c = 0\}| \\ &= |\{(x_1, x_2) \in \mathbb{F}_q^2 : a_1 x_1^c + a_2 x_2^c + a_3 (m_1' x_1 + m_2' x_2)^c = 0\}| \lesssim q, \end{aligned}$$

where the last inequality follows from the Schwartz-Zippel lemma, because one can check that $a_1 x_1^c + a_2 x_2^c + a_3 (m_1' x_1 + m_2' x_2)^c$ is a nonzero polynomial for $c \geq 2$ and $a_j \neq 0, j = 1,2,3$. Thus, we complete the proof of Theorem 6.3.

7. Note on problems for homogeneous varieties in higher odd dimensions

We have seen that the Fourier decay estimates on homogeneous varieties make a key role to study the problems in this paper. We have been mentioning that one could obtain good Fourier decay estimates on homogeneous varieties in odd dimensions, but fail in even dimensions. For reader's convenience, we give the following explicit computations to indicate the estimates of the Fourier transform on quadratic homogeneous varieties are different between odd and even dimensions. Now let us see the following two examples. First, suppose that $V = \{x \in \mathbb{F}_q^3 : x_1^2 + x_2^2 + x_3^2 = 0\}$. Then, for each $m \neq (0,0,0)$, we have

$$\widehat{V}(m) = q^{-4} \sum_{s \neq 0} \prod_{j=1}^{3} \sum_{x_j \in \mathbb{F}_q} \chi(s x_j^2 - m_j x_j).$$

Completing the square and making a change of variables, we observe that

$$\sum_{x_j \in \mathbb{F}_q} \chi(sx_j^2 - m_j x_j) = \sum_{x_j \in \mathbb{F}_q} \chi(sx_j^2)\chi\left(\frac{m_j^2}{-4s}\right)$$
$$= G\eta(s)\chi\left(\frac{m_j^2}{-4s}\right),$$

where G denotes the Gauss sum, η denotes the quadratic character, and we use the fact that $\sum_{x_j \in \mathbb{F}_q} \chi(sx_j^2) = G\eta(s)$. Thus, we see that for $m \neq (0,0,0)$,

$$\widehat{V}(m) = q^{-4}G^3 \sum_{s \neq 0} \eta^3(s)\chi\left(\frac{m_1^2 + m_2^2 + m_3^2}{-4s}\right).$$

Since η is the quadratic character, $\eta^3 = \eta$, and so the sum over $s \neq 0$ is a Salié sum [20] which is always $\lesssim q^{1/2}$. Thus, we get a good Fourier decay estimate on V for all $m \neq (0,0,0)$. Namely, we have for $m \neq (0,0,0)$,

$$|\widehat{V}(m)| \lesssim q^{-2} = q^{-\frac{d+1}{2}},$$

which is what we want.

However, now consider $V = \{x \in \mathbb{F}_q^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0\}$. Using above method, we see that for each $m \neq (0,0,0,0)$

$$\widehat{V}(m) = q^{-5}G^4 \sum_{s \neq 0} \eta^4(s)\chi\left(\frac{m_1^2 + m_2^2 + m_3^2 + m_4^2}{-4s}\right).$$

Since $\eta^4 = 1$, the sum over $s \neq 0$ is $(q-1)$ if $m_1^2 + m_2^2 + m_3^2 + m_4^2 = 0$. Thus, for some $m \neq (0,0,0,0)$, we have

$$|\widehat{V}(m)| \sim q^{-2} = q^{-\frac{d}{2}},$$

which is worse than $q^{-\frac{5}{2}} = q^{-\frac{d+1}{2}}$. Therefore, the question we want to address first is whether the estimate (6.3) in Lemma 6.5 can be improved in all odd dimensions. If the dimension is even, then above estimates say that we can not expect to improve it (at least for the case $s = 2$). However, we already observed that in three dimension the estimate (6.3) can be improved to $|\widehat{H_0}(m)| \lesssim q^{-\frac{d+1}{2}} = q^{-2}$ for $m \neq (0,0,0)$. From these facts, one may have the following question.

Question 7.1. *Let* $P(x) = \sum_{j=1}^{d} a_j x_j^s \in \mathbb{F}_q[x_1, \ldots, x_d]$ *with* $s \geq 2, a_j \neq 0$ *for all* $j = 1, \ldots, d$. *If we assume that the characteristic of* $\mathbb{F}_q$ *is sufficiently large and the dimension* $d \geq 3$ *is odd, then does the following conclusion always hold?*

$$|\widehat{H}(m)| \lesssim q^{-\frac{d+1}{2}} \quad \text{for all } m \in \mathbb{F}_q^d \setminus \{(0,\ldots,0)\},$$

where $H = \{x \in \mathbb{F}_q^d : P(x) = 0\}$.

If the answer to Question 7.1 is positive, then this would yield the standard Tomas-Stein exponents on the extension problem related to diagonal polynomials in odd dimensions. Moreover, the averaging problem on homogeneous varieties in odd dimensions would be completely understood. As a trial to find the answer to Question 7.1, one may invoke some powerful results from algebraic geometry such as [13] and [4]. However, it seems that such theorems do not explain that the Fourier decays of homogeneous varieties in odd dimensions are better than those in even dimensions. We close this paper with a desire to see the answer to Question 7.1 in the near future.

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