# Cardinalities of distance sets determined by two sets in vector spaces over finite fields

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Abstract. In this paper we investigate the size of the distance set determined by arbitrary two sets in vector spaces over finite fields. We estimate a lower bound of the distance set which is explicitly expressed in terms of cardinalities of two sets. As a result, we improve upon the results by Rainer Dietmann [**3**]. In particular, in the case when one of the sets is a product set, we obtain further improvement on the cardinality of the distance set.

## 1. Introduction

Let $E, F$ be finite point sets in $\mathbb{R}^d$. The distance set determined by $E$ and $F$ is defined by

$$\Delta(E,F) = \{|x-y| : x \in E, y \in F\},$$

where $|\cdot|$ denotes the standard norm on $\mathbb{R}^d$. The main problem is to find the cardinality of the distance set $\Delta(E,F)$. In the case when $E = F$, Erdős [**4**] first addressed this problem and showed that

$$|\Delta(E,E)| \geq c|E|^{\frac{1}{d}}$$

where $|E|$ denotes the number of elements in $E$ and $c > 0$ is independent of the set $E \subset \mathbb{R}^d$. Taking the set $E$ as a piece of the integer lattice, the Erdős distance conjecture says that for every $\varepsilon > 0$, there exists a $c_\varepsilon > 0$ such that

$$|\Delta(E,E)| \geq c_\varepsilon |E|^{\frac{2}{d}-\varepsilon}.$$

In dimension two, the conjecture has recently been solved by Guth and Katz [**5**], who proved that

$$|\Delta(E,E)| \geq c\frac{|E|}{\log|E|}.$$

However, the Erdős distance conjecture is still open for higher dimensions. See [**12**], [**16**], [**17**], and the references contained therein for recent developments on the Erdős distance problem in higher dimensions.

As an analog of the Euclidean Erdős distance problem, Bourgain,Katz, and Tao [**1**] posed and studied the finite field version of the Erdős distance problem in two dimensions. The Erdős distance problem in the finite field setting has been recently studied by several people

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(see [**6**], [**18**], [**19**],[**20**], [**2**], [**10**], and [**11**]). Let $\mathbb{F}_q^d$ be a $d$-dimensional vector space over the finite field $\mathbb{F}_q$ with $q$ elements. We shall always assume that the characteristic of $\mathbb{F}_q$ is greater than two. In the finite field setting, given two sets $E, F \subset \mathbb{F}_q^d$, the distance set may be defined by

$$\Delta(E,F) = \{\|x-y\| \in \mathbb{F}_q : x \in E, y \in F\},$$

where we define $\|m\| = m_1^2 + \cdots + m_d^2$ for $m = (m_1, \ldots, m_d) \in \mathbb{F}_q^d$. Assuming that $E \subset \mathbb{F}_q^2$ with prime $q \equiv 3 (\mathrm{mod}\ 4)$, the aforementioned authors [**1**] proved that if $|E| \le q^{2-\varepsilon}$ for some $\varepsilon > 0$, then there exists a $\delta(\varepsilon) > 0$ such that $|\Delta(E,E)| \ge c|E|^{\frac{1}{2}+\delta(\varepsilon)}$. However, the value $\delta(\varepsilon)$ was not given in an explicit form. In [**9**], Iosevich and Rudnev obtained distance set results on general finite fields in arbitrary dimensions with explicit exponents. More precisely, they proved that if $E \subset \mathbb{F}_q^d$ and $|E| \ge Cq^{\frac{d}{2}}$ with a sufficiently large $C > 0$, then

$$|\Delta(E,E)| \ge c \min\left\{q, \frac{|E|}{q^{\frac{d-1}{2}}}\right\}.$$

In [**15**], Shparlinski derived an explicit lower bound of the number of the distances between arbitrary two sets: if $E, F \subset \mathbb{F}_q^d$, then

$$(1.1) \qquad |\Delta(E,F)| > \frac{|E||F|q}{q^{d+1} + |E||F|} \ge \frac{1}{2} \min\left\{q, \frac{|E||F|}{q^d}\right\}.$$

Dietmann [**3**] recently obtained a new lower bound of $|\Delta(E,F)|$. In fact, he proved that if $E, F \subset \mathbb{F}_q^d$, $|F| \ge |E|$, and $|E||F| \ge (900 + \log q)q^d$, then

$$(1.2) \qquad |\Delta(E,F)| \ge \begin{cases} c\min\left\{q, \frac{|F|}{q^{\frac{d-1}{2}}\log q}\right\} & \text{for } d \ge 2 \\ c\min\left\{q, \frac{|E|^{\frac{1}{2}}|F|}{q\log q}\right\} & \text{for } d = 2 \end{cases}.$$

Shparlinski's result (1.1) gives an explicit lower bound of $|\Delta(E,F)|$ but it is not stronger than Dietmann's result (1.2) if $d = 2$ or $|E|$ is much smaller than $q^{(d+1)/2}(\log q)^{-1}$. On the other hand, Dietmann's theorem contains a $\log q$ factor and the strong assumption that $|E||F| \ge (900 + \log q)q^d$. Note that both results of Shparlinski and Dietmann are nontrivial only if $|E||F| > q^d$. In fact, if $d \ge 2$ is even and $-1 \in \mathbb{F}_q$ is a square number, then there exist $E, F \subset \mathbb{F}_q^d$ such that $|E||F| = q^d$ and $|\Delta(E,F)| = 1$. To see this, if $d \ge 2$ is even and $i^2 = -1$ for some $i \in \mathbb{F}_q$, then choose

$$(1.3) \qquad E = F = \left\{(t_1, it_1, \ldots, t_j, it_j, \ldots, t_{d/2}, it_{d/2}) \in \mathbb{F}_q^d : t_j \in \mathbb{F}_q, j = 1, \ldots, d/2\right\}.$$

However, it turns out that if $d \ge 3$ is odd, then such an example does not exist and one may expect that $|\Delta(E,F)| > 1$ for $|E||F| > q^{d-1}$. For this reason, if $d \ge 3$ is odd and $q^{d-1} < |E||F| \le q^d$, then we must have a meaningful lower bound of $|\Delta(E,F)|$ other than trivial one. In this paper we will clarify this point. In addition, we shall observe that the $\log q$ factor can be eliminated in Dietmann's result (1.2). Finally, we study the number of distances between a product set and arbitrary set in $\mathbb{F}_q^d$.

## 2. Statement of main results

THEOREM 2.1. *If $d \geq 3$ is odd, and $E, F \subset \mathbb{F}_q^d$, then*

$$|\Delta(E,F)| \geq \begin{cases} \min\left\{\frac{q}{2}, \frac{|E||F|}{8q^{d-1}}\right\} & \text{if } 1 \leq |E| < q^{\frac{d-1}{2}} \\ \min\left\{\frac{q}{2}, \frac{|F|}{8q^{\frac{d-1}{2}}}\right\} & \text{if } q^{\frac{d-1}{2}} \leq |E| < q^{\frac{d+1}{2}} \\ \min\left\{\frac{q}{2}, \frac{|E||F|}{2q^{d}}\right\} & \text{if } q^{\frac{d+1}{2}} \leq |E| \leq q^d \end{cases}.$$

Since $\Delta(E,F) = \Delta(F,E)$, we may assume that $|E| \leq |F|$ in the statement of Theorem 2.1. We claim that Theorem 2.1 is much better than Dietmann's result (1.2) for odd dimensions. To see this, first notice that our claim follows immediately from a direct comparison if $q^{\frac{d-1}{2}} \leq |E| \leq q^d$. In this case, Theorem 2.1 says that the $\log q$ in Dietmann's result can be removed. Next, we see from Theorem 2.1 that if $1 \leq |E| < q^{\frac{d-1}{2}}$ and $|E||F| \geq 4q^d$, then $|\Delta(E,F)| \geq \frac{q}{2}$ for $d \geq 3$ odd, which is also better than Diemann's result. Now, let us compare Theorem 2.1 with Shparlinski's result (1.1). If $d \geq 3$ is odd and $1 \leq |E| < q^{\frac{d+1}{2}}$, then Theorem 2.1 is clearly stronger than Shparlinski's result. One important point is that if $|E||F| \leq q^d$, then Shparlinski's result says noting more than $|\Delta(E,F)| = 1$. The same thing can be said for Dietmann's result, because his result depends on a strong assumption that $|E||F| \geq (900 + \log q)q^d$. In contrast, Theorem 2.1 gives a meaningful information about $|\Delta(E,F)|$ whenever $d \geq 3$ is odd and $8q^{d-1} < |E||F| \leq q^d$. For example, if $d \geq 3$ is odd, $|E| = q^{(d-1)/2} - 1$, and $|F| = q^{(d+1)/2}$, then $|E||F| = q^d - q^{(d+1)/2} < q^d$, but $|\Delta(E,F)| \geq \frac{q}{12}$.

THEOREM 2.2. *Let $d \geq 2$ be even and $E, F \subset \mathbb{F}_q^d$. If $|E||F| \geq 16q^d$, then we have*

$$|\Delta(E,F)| \geq \begin{cases} \frac{q}{144} & \text{for } |E| < q^{\frac{d-1}{2}} \\ \frac{1}{144}\min\left\{q, \frac{|F|}{2q^{\frac{d-1}{2}}}\right\} & \text{for } q^{\frac{d-1}{2}} \leq |E| < q^{\frac{d+1}{2}} \\ \frac{1}{144}\min\left\{q, \frac{2|E||F|}{q^{d}}\right\} & \text{for } q^{\frac{d+1}{2}} \leq |E| \leq q^d. \end{cases}$$

*In addition, if $d = 2$ and $|E||F| \geq 16q^2$, then*

$$|\Delta(E,F)| \geq \frac{1}{72}\min\left\{\frac{q}{2}, \frac{|E|^{\frac{1}{2}}|F|}{\sqrt{3}q}\right\}.$$

*In particular, if $d = 2$ and $-1 \in \mathbb{F}_q$ is not a square number, then*

$$|\Delta(E,F)| \geq \min\left\{\frac{q}{2}, \frac{|E|^{\frac{1}{2}}|F|}{2(\sqrt{3}+1)q}\right\}.$$

Theorem 2.2 is better than Dietmann's result (1.2) for even dimensions in that the $\log q$ is not necessary in the results. In particular, we see that $|\Delta(E,F)| \geq \frac{q}{144}$ if $|E| < q^{(d-1)/2}$ and $|E||F| = 16q^d$. In dimension two, the second part of Theorem 2.2 enables us to improve the first part of Theorem 2.2 in the case when $q \leq |E| \leq q^2$. As mentioned above, if $d \geq 3$ is odd, then $|\Delta(E,F)| > 1$ whenever $8q^{d-1} < |E||F| \leq q^d$. However, this is not true any more in even dimensions as observed in the example (1.3). For this reason, we need the assumption that $|E||F| \geq 16q^d$ in Theorem 2.2. The last part of Theorem 2.2 says that if we assume that $d = 2$ and $-1 \in \mathbb{F}_q$ is not a square number, then we can drop the assumption in the second part of Theorem 2.2 that $|E||F| \geq 16q^2$. In this case, we have nontrivial distance results whenever $|E|^{\frac{1}{2}}|F| > 2(\sqrt{3}+1)q$.

THEOREM 2.3. *Let $d \geq 2$. Suppose that $E = A \times A \times \cdots \times A \subset \mathbb{F}_q^d$ is a product set and $F \subset \mathbb{F}_q^d$. Then we have*

$$|\Delta(E,F)| \geq \min\left\{\frac{q}{2}, \frac{|E|^{1-\frac{1}{d}}|F|}{4q^{d-1}}\right\}. \tag{2.1}$$

Notice that if $q^{d/2} \leq |E| \leq q^d/2^d$, then the conclusion of Theorem 2.3 is superior to the conclusions of both Theorem 2.1 and 2.2. In particular, the conclusion of Theorem 2.3 holds true without the assumption in Theorem 2.2 that $|E||F| \geq 16q^d$. In fact, it yields nontrivial results whenever $|E|^{1-\frac{1}{d}}|F| > 4q^{d-1}$, a weaker condition than $|E||F| \geq 16q^d$.

REMARK 2.4. As we shall see in the proof of Theorem 2.3, the assumption that $E = A \times A \times \cdots \times A$ can be replaced by a weaker condition that $E = \underline{E} \times A \subset \mathbb{F}_q^{d-1} \times \mathbb{F}_q$ and $|\underline{E}| = |A|^{d-1}$. In particular, if we assume that $E = \underline{E} \times A \subset \mathbb{F}_q^{d-1} \times \mathbb{F}_q$ and $|\underline{E}| > |A|^{d-1}$, then we could obtain much stronger conclusion than (2.1).

## 3. Discrete Fourier analysis and lemmas

Iosevich and Rudnev [**9**] adapted the discrete Fourier analysis to measure the size of distance sets in the finite field setting. As a result, they developed a powerful machinery for deriving results on the Erdős distance problem. In this section, we review it and obtain more accurate formula for the number of distances between two different sets. We begin with the definition of the Fourier tansform. Given a function $f : \mathbb{F}_q^d \to \mathbb{C}$, the Fourier transform of $f$ is given by the form

$$\widehat{f}(m) = \frac{1}{q^d}\sum_{x \in \mathbb{F}_q^d} \chi(-m\cdot x) f(x) \quad \text{for } m \in \mathbb{F}_q^d.$$

Here, and throughout this paper, we denote by $\chi$ a fixed nontrivial additive character of $\mathbb{F}_q$ and the results on the distance problems will be independent of the choice of the character. Recall that the orthogonality relation of $\chi$ says that

$$\sum_{x \in \mathbb{F}_q^d} \chi(m \cdot x) = \begin{cases} 0 & \text{if } m \neq (0,\ldots,0) \\ q^d & \text{if } m = (0,\ldots,0) \end{cases}.$$

The following Fourier inversion formula follows immediately from a direct application of the orthogonality relation of $\chi$:

$$f(x) = \sum_{m \in \mathbb{F}_q^d} \chi(m\cdot x)\widehat{f}(m).$$

Moreover, the Plancherel theorem yields the following formula:

$$\sum_{m \in \mathbb{F}_q^d} |\widehat{f}(m)|^2 = q^{-d}\sum_{x \in \mathbb{F}_q^d} |f(x)|^2.$$

Notice from the Plancherel theorem that if $E \subset \mathbb{F}_q^d$, then

$$\sum_{m \in \mathbb{F}_q^d} |\widehat{E}(m)|^2 = q^{-d}\sum_{x \in \mathbb{F}_q^d} |E(x)|^2 = q^{-d}|E|.$$

Here, and throughout this paper, we identify the set $E$ with the characteristic function $\chi_E$ on the set $E$.

Let us denote by $G, K, S$ the Gauss sum, the Kloosterman sum, and the Salié sum, respectively. Recall that for $a, b \in \mathbb{F}_q^*$,

$$G = \sum_{s\in\mathbb{F}_q^*} \eta(s)\chi(s), \quad K = \sum_{s\in\mathbb{F}_q^*} \chi(as+bs^{-1}), \quad \text{and} \quad S = \sum_{s\in\mathbb{F}_q^*} \eta(s)\chi(as+bs^{-1}),$$

where $\eta$ denotes the quadratic character of $\mathbb{F}_q^* := \mathbb{F}_q \setminus \{0\}$. It is well known that

$$|G| = \sqrt{q}, \ |K| \le 2\sqrt{q}, \ \text{and} \ |S| \le 2\sqrt{q}. \tag{3.1}$$

For a nice proof of these exponential sum estimates, see p.193 in [**13**] and pp.322-323 in [**8**]. For each $t \in \mathbb{F}_q$, we define a sphere with $t$ radius as the set

$$S_t = \{x \in \mathbb{F}_q^d : \|x\| = t\}.$$

The Fourier transform on $S_t$ is closely related to aforementioned exponential sums. It was proved in [**7**] that for $t \in \mathbb{F}_q, m \in \mathbb{F}_q^d$,

$$\widehat{S}_t(m) = q^{-1}\delta_0(m) + q^{-d-1}\eta^d(-1)G^d \sum_{r\in\mathbb{F}_q^*} \eta^d(r)\chi\Big(tr + \frac{\|m\|}{4r}\Big), \tag{3.2}$$

where $\delta_0(m) = 1$ if $m = (0, \dots, 0)$ and $\delta_0(m) = 0$ otherwise. The following corollary is an immediate consequence from (3.2) and (3.1).

Corollary 3.1. *If $d \ge 3$ is odd, then we have*

$$|\widehat{S}_t(m)| \le 2q^{-\frac{d+1}{2}} \quad \textit{for } t \in \mathbb{F}_q, m \in \mathbb{F}_q^d \setminus \{(0,\dots,0)\},$$

*and*

$$\widehat{S}_t(0,\dots,0) = q^{-d}|S_t| \le 2q^{-1} \quad \textit{for } t \in \mathbb{F}_q.$$

*On the other hand, if $d \ge 2$ is even, then we have*

$$|\widehat{S}_t(m)| \le 2q^{-\frac{d+1}{2}} \quad \textit{for } t \in \mathbb{F}_q^*, \ m \in \mathbb{F}_q^d \setminus \{(0,\dots,0)\},$$

$$|\widehat{S}_0(m)| \le q^{-\frac{d}{2}} \quad \textit{for } m \in S_0 \setminus \{(0,\dots,0)\},$$

*and*

$$\widehat{S}_t(0,\dots,0) = q^{-d}|S_t| \le 2q^{-1} \quad \textit{for } t \in \mathbb{F}_q.$$

The following lemma can be obtained by applying the orthogonality relation of $\chi$.

Lemma 3.2. *For $m, m' \in \mathbb{F}_q^d$, we have*

$$\sum_{t\in\mathbb{F}_q} \widehat{S}_t(m)\overline{\widehat{S}_t}(m') = q^{-1}\delta_0(m)\delta_0(m') + q^{-d-1}\sum_{s\in\mathbb{F}_q^*} \chi(s(\|m\| - \|m'\|)).$$

Proof. For $m \in \mathbb{F}_q^d, t \in \mathbb{F}_q$, write $\widehat{S}_t(m) = q^{-1}\delta_0(m) + R_t(m)$, where $R_t(m)$ denotes the second term of the right-hand side in (3.2). It follows that for $m, m' \in \mathbb{F}_q^d$,

$$\sum_{t\in\mathbb{F}_q} \widehat{S}_t(m)\overline{\widehat{S}_t}(m') = \sum_{t\in\mathbb{F}_q} \left(q^{-1}\delta_0(m) + R_t(m)\right)\left(q^{-1}\delta_0(m') + \overline{R_t}(m')\right)$$

$$= \sum_{t\in\mathbb{F}_q} q^{-2}\delta_0(m)\delta_0(m') + q^{-1}\delta_0(m)\sum_{t\in\mathbb{F}_q}\overline{R_t}(m') + q^{-1}\delta_0(m')\sum_{t\in\mathbb{F}_q}R_t(m) + \sum_{t\in\mathbb{F}_q}R_t(m)\overline{R_t}(m').$$

By the orthogonality relation of $\chi$, the sums in the second and third terms vanish. Thus, we have

$$\sum_{t\in\mathbb{F}_q}\widehat{S}_t(m)\overline{\widehat{S}_t}(m') = q^{-1}\delta_0(m)\delta_0(m') + q^{-d-2}\sum_{s,s'\in\mathbb{F}_q^*}\eta^d(s)\overline{\eta}^d(s')\chi\left(\frac{\|m\|}{4s} - \frac{\|m'\|}{4s'}\right)\sum_{t\in\mathbb{F}_q}\chi((s-s')t)$$

$$= q^{-1}\delta_0(m)\delta_0(m') + q^{-d-1}\sum_{s\in\mathbb{F}_q^*}\chi\left(\frac{\|m\|-\|m'\|}{4s}\right)$$

$$= q^{-1}\delta_0(m)\delta_0(m') + q^{-d-1}\sum_{s\in\mathbb{F}_q^*}\chi\left(s(\|m\|-\|m'\|\right).$$

□

Now, we review standard distance formulas which were originally due to Iosevich and Rudenv [**9**]. Given two sets $E, F \subset \mathbb{F}_q^d$, we consider a counting function $\nu : \mathbb{F}_q \to \mathbb{N}\cup\{0\}$ defined by

$$\nu(t) = |\{(x,y)\in E\times F : \|x-y\| = t\}| \quad \text{for } t\in\mathbb{F}_q.$$

The distance formulas are as follows.

Lemma 3.3. *Let $E, F \subset \mathbb{F}_q^d, d \geq 2$. Then we have*

$$|\Delta(E,F)| \geq \frac{|E|^2|F|^2}{\sum_{t\in\mathbb{F}_q}\nu^2(t)} \tag{3.3}$$

*and*

$$|\Delta(E,F)| \geq \frac{(|E||F|-\nu(0))^2}{\sum_{t\in\mathbb{F}_q^*}\nu^2(t)}. \tag{3.4}$$

Proof. Since $|E||F| = \sum_{t\in\mathbb{F}_q}\nu(t)$ and $|E||F|-\nu(0) = \sum_{t\in\mathbb{F}_q^*}\nu(t)$, the Cauchy-Schwarz inequality yields

$$|E|^2|F|^2 = \left(\sum_{t\in\mathbb{F}_q}\nu(t)\right)^2 \leq |\Delta(E,F)|\sum_{t\in\mathbb{F}_q}\nu^2(t),$$

and

$$(|E||F|-\nu(0))^2 = \left(\sum_{t\in\mathbb{F}_q^*}\nu(t)\right)^2 \leq |\Delta(E,F)|\sum_{t\in\mathbb{F}_q^*}\nu^2(t).$$

Thus, (3.3) and (3.4) follow immediately from these observations. □

Remark 3.4. As we shall see, the inequality (3.3) is used to prove our distance results in odd dimensions. On the other hand, the inequality (3.4) is useful in even dimensional case. Iosevich and Rudnev [**9**] and Dietmann [**3**] made use of the formula (3.4) to derive distance results. Consequently, they obtained the nontrivial distance results in the case when $|E||F| \geq Cq^d$. In this paper we point out that if the dimension $d\geq 3$ is odd, then the formula (3.3) enables us to yield nontrivial results whenever $|E||F| \geq Cq^{d-1}$. In [**3**], Dietmann obtained the result in (1.2) by estimating $\sum_{t\in\mathbb{F}_q^*}\nu^2(t)$. To the end, he applied the pigeonhole principle so that his result contains the $\log q$ factor. In this paper we observe that the $\log q$ factor can be eliminated by a direct estimate rather than the pigeonhole principle.

The $l^2$ estimate of the counting function $\nu$ takes the following form.

LEMMA 3.5.

$$\sum_{t\in\mathbb{F}_q}\nu^2(t)\le q^{-1}|E|^2|F|^2+q^{2d}|F|\left(\max_{r\in\mathbb{F}_q}\sum_{m\in S_r}|\widehat{E}(m)|^2\right) \tag{3.5}$$

*and*

$$\sum_{t\in\mathbb{F}_q}\nu^2(t)\le q^{-1}|E|^2|F|^2+q^{3d}\left|\sum_{m\in S_0}\overline{\widehat{E}}(m)\widehat{F}(m)\right|^2+q^{2d}|F|\left(\max_{r\in\mathbb{F}_q^*}\sum_{m\in S_r}|\widehat{E}(m)|^2\right). \tag{3.6}$$

PROOF. For each $t\in\mathbb{F}_q$, we have

$$\nu(t)=\sum_{x\in E,y\in F}S_t(x-y)=\sum_{x,y\in\mathbb{F}_q^d}E(x)F(y)\sum_{m\in\mathbb{F}_q^d}\chi(m\cdot(x-y))\widehat{S}_t(m),$$

where the Fourier inversion formula was applied to $S_t(x-y)$. By the definition of the Fourier tansform, we see that

$$\nu(t)=q^{2d}\sum_{m\in\mathbb{F}_q^d}\widehat{S}_t(m)\overline{\widehat{E}}(m)\widehat{F}(m). \tag{3.7}$$

Squaring $\nu(t)$ and summing over $t\in\mathbb{F}_q$, it follows from Lemma 3.2 that

$$\sum_{t\in\mathbb{F}_q^d}\nu^2(t)=q^{4d}\sum_{m,m'\in\mathbb{F}_q^d}\overline{\widehat{E}}(m)\widehat{F}(m)\widehat{E}(m')\overline{\widehat{F}}(m')\sum_{t\in\mathbb{F}_q}\widehat{S}_t(m)\overline{\widehat{S}_t}(m')$$

$$=q^{4d}\sum_{m,m'\in\mathbb{F}_q^d}\overline{\widehat{E}}(m)\widehat{F}(m)\widehat{E}(m')\overline{\widehat{F}}(m')\left(q^{-1}\delta_0(m)\delta_0(m')+q^{-d-1}\sum_{s\in\mathbb{F}_q}\chi(s(\|m\|-\|m'\|)-q^{-d-1}\right)$$

$$=\frac{|E|^2|F|^2}{q}+q^{3d-1}\sum_{m,m'\in\mathbb{F}_q^d}\overline{\widehat{E}}(m)\widehat{F}(m)\widehat{E}(m')\overline{\widehat{F}}(m')\sum_{s\in\mathbb{F}_q}\chi(s(\|m\|-\|m'\|)$$

$$-q^{3d-1}|\sum_{m\in\mathbb{F}_q^d}\overline{\widehat{E}}(m)\widehat{F}(m)\ |^2.$$

Notice that the last term is negative. Applying the orthogonality relation of $\chi$ to the sum over $s$ in the second term above, we obtain that

$$\sum_{t\in\mathbb{F}_q^d}\nu^2(t)\le\frac{|E|^2|F|^2}{q}+q^{3d}\sum_{m,m'\in\mathbb{F}_q^d:\|m\|=\|m'\|}\overline{\widehat{E}}(m)\widehat{F}(m)\widehat{E}(m')\overline{\widehat{F}}(m').$$

This can be written by

$$\sum_{t\in\mathbb{F}_q^d}\nu^2(t)\le\frac{|E|^2|F|^2}{q}+q^{3d}\sum_{r\in\mathbb{F}_q}\left|\sum_{m\in S_r}\overline{\widehat{E}}(m)\widehat{F}(m)\right|^2. \tag{3.8}$$

The inequality (3.5) follows from the Cauchy-Schwarz inequality and the Plancherel theorem, because

$$\begin{aligned}\sum_{t\in\mathbb{F}_q^d}\nu^2(t) &\leq \frac{|E|^2|F|^2}{q}+q^{3d}\sum_{r\in\mathbb{F}_q}\left(\sum_{m\in S_r}|\widehat{E}(m)|^2\right)\left(\sum_{m\in S_r}|\widehat{F}(m)|^2\right)\\ &\leq \frac{|E|^2|F|^2}{q}+q^{3d}\left(\max_{r\in\mathbb{F}_q}\sum_{m\in S_r}|\widehat{E}(m)|^2\right)\sum_{m\in\mathbb{F}_q^d}|\widehat{F}(m)|^2\\ &= q^{-1}|E|^2|F|^2+q^{2d}|F|\left(\max_{r\in\mathbb{F}_q}\sum_{m\in S_r}|\widehat{E}(m)|^2\right).\end{aligned}$$

In order to prove the inequality (3.6), first notice from (3.8) that

$$\sum_{t\in\mathbb{F}_q^d}\nu^2(t)\leq \frac{|E|^2|F|^2}{q}+q^{3d}\left|\sum_{m\in S_0}\overline{\widehat{E}}(m)\widehat{F}(m)\right|^2+q^{3d}\sum_{r\in\mathbb{F}_q^*}\left|\sum_{m\in S_r}\overline{\widehat{E}}(m)\widehat{F}(m)\right|^2$$

and estimate the third term by the previous arguments. Then the inequality (3.6) follows immediately. □

The following lemma plays a crucial role in proving results in two dimensions and it was proved in [**2**] (see Lemma 4.4 in [**2**].)

LEMMA 3.6. *If $E\subset\mathbb{F}_q^2$, then*

$$\max_{r\in\mathbb{F}_q^*}\sum_{m\in S_r}|\widehat{E}(m)|^2\leq\sqrt{3}q^{-3}|E|^{\frac{3}{2}}.$$

## 4. Proof of Theorem 2.1

Combining (3.3) in Lemma 3.3 with (3.5) in Lemma 3.5, we see that

$$|\Delta(E,F)|\geq\frac{|E|^2|F|^2}{q^{-1}|E|^2|F|^2+q^{2d}|F|\left(\max_{r\in\mathbb{F}_q}\sum_{m\in S_r}|\widehat{E}(m)|^2\right)}. \tag{4.1}$$

Then our task is to find a good upper bound of $\max_{r\in\mathbb{F}_q}\sum_{m\in S_r}|\widehat{E}(m)|^2$. Suppose that $d\geq 3$ is odd. For each $r\in\mathbb{F}_q$, the Plancherel theorem yields $\sum_{m\in S_r}|\widehat{E}(m)|^2\leq\sum_{m\in\mathbb{F}_q^d}|\widehat{E}(m)|^2=q^{-d}|E|$. Hence, we have

$$\max_{r\in\mathbb{F}_q}\sum_{m\in S_r}|\widehat{E}(m)|^2\leq q^{-d}|E|. \tag{4.2}$$

Next, from the definition of the Fourier transform, it follows that for each $r\in\mathbb{F}_q$,

$$\sum_{m\in S_r}|\widehat{E}(m)|^2=q^{-d}\sum_{x,y\in E}\widehat{S_r}(x-y)=q^{-d}|E|\widehat{S_r}(0,\ldots,0)+q^{-d}\sum_{x,y\in E:x-y\neq(0,\ldots,0)}\widehat{S_r}(x-y).$$

Since $d\geq 3$ is odd, we see from the first part of Corollary 3.1 that

$$\max_{r\in\mathbb{F}_q}\sum_{m\in S_r}|\widehat{E}(m)|^2\leq 2q^{-d-1}|E|+2q^{-\frac{3d+1}{2}}|E|^2.$$

This estimate and (4.2) yield that if $d \geq 3$ is odd, then

$$\max_{r\in\mathbb{F}_q}\sum_{m\in S_r}|\widehat{E}(m)|^2 \leq \min\{q^{-d}|E|,\ 2q^{-d-1}|E|+2q^{-\frac{3d+1}{2}}|E|^2\}$$

$$\leq \begin{cases} 4q^{-d-1}|E| & \text{if } 1\leq |E| < q^{\frac{d-1}{2}} \\ 4q^{-\frac{3d+1}{2}}|E|^2 & \text{if } q^{\frac{d-1}{2}} \leq |E| < q^{\frac{d+1}{2}} \\ q^{-d}|E| & \text{if } q^{\frac{d+1}{2}} \leq |E| \leq q^d. \end{cases}$$

By the inequality (4.1) and this estimate, a direct computation shows that if $d \geq 3$ is odd, then

$$|\Delta(E,F)| \geq \begin{cases} \min\left\{\frac{q}{2}, \frac{|E||F|}{8q^{d-1}}\right\} & \text{if } 1\leq |E| < q^{\frac{d-1}{2}} \\ \min\left\{\frac{q}{2}, \frac{|F|}{8q^{\frac{d-1}{2}}}\right\} & \text{if } q^{\frac{d-1}{2}} \leq |E| < q^{\frac{d+1}{2}} \\ \min\left\{\frac{q}{2}, \frac{|E||F|}{2q^{d}}\right\} & \text{if } q^{\frac{d+1}{2}} \leq |E| \leq q^d. \end{cases}$$

Thus, the proof of Theorem 2.1 is complete.

## 5. proof of Theorem 2.2

For $d \geq 2$, (3.4) in Lemma 3.3 and (3.6) in Lemma 3.5 imply that

(5.1)

$$|\Delta(E,F)| \geq \frac{(|E||F|-\nu(0))^2}{q^{-1}|E|^2|F|^2 + q^{2d}|F|\left(\max\limits_{r\in\mathbb{F}_q^*}\sum\limits_{m\in S_r}|\widehat{E}(m)|^2\right) + q^{3d}\left|\sum\limits_{m\in S_0}\overline{\widehat{E}}(m)\widehat{F}(m)\right|^2 - \nu^2(0)}.$$

### 5.1. Lemmas for the proof of Theorem 2.2.

We need the following estimate.

Lemma 5.1. *Let $E, F \subset \mathbb{F}_q^d$. If $d \geq 2$ is even, then we have*

$$\nu(0) = q^dG^d\sum_{m\in S_0}\overline{\widehat{E}}(m)\widehat{F}(m) + M(E,F), \tag{5.2}$$

*where $M(E,F) = q^{-1}|E||F| - q^{d-1}G^d\sum_{m\in\mathbb{F}_q^d}\overline{\widehat{E}}(m)\widehat{F}(m)$. In addition, we have*

$$|\nu(0)| \leq q^{-1}|E||F| + 2q^{\frac{d}{2}}|E|^{\frac{1}{2}}|F|^{\frac{1}{2}}. \tag{5.3}$$

Proof. Since $d \geq 2$ is even, it follows from (3.7) and (3.2) that

$$\nu(0) = q^{2d}\sum_{m\in\mathbb{F}_q^d}\overline{\widehat{E}}(m)\widehat{F}(m)\left(q^{-1}\delta_0(m) + q^{-d-1}G^d\sum_{r\in\mathbb{F}_q^*}\chi\left(\frac{\|m\|}{4r}\right)\right)$$

$$= q^{-1}|E||F| + q^{d-1}G^d\sum_{m\in\mathbb{F}_q^d}\overline{\widehat{E}}(m)\widehat{F}(m)\left(\sum_{s\in\mathbb{F}_q}\chi(s\|m\|)-1\right)$$

$$= q^{-1}|E||F| + q^dG^d\sum_{m\in S_0}\overline{\widehat{E}}(m)\widehat{F}(m) - q^{d-1}G^d\sum_{m\in\mathbb{F}_q^d}\overline{\widehat{E}}(m)\widehat{F}(m).$$

This completes the proof of (5.2). To prove (5.3), notice from (5.2) that

$$|\nu(0)| \le q^{-1}|E||F| + 2|q^d G^d| \sum_{m\in\mathbb{F}_q^d} |\overline{\widehat{E}}(m)|\ |\widehat{F}(m)|.$$

Using the Gauss sum estimate, the Cauchy-Schwarz inequality, and the Plancherel theorem, we conclude that

$$|\nu(0)| \le q^{-1}|E||F| + 2q^d q^{\frac{d}{2}} \left(\sum_{m\in\mathbb{F}_q^d} |\widehat{E}(m)|^2\right)^{\frac{1}{2}} \left(\sum_{m\in\mathbb{F}_q^d} |\widehat{F}(m)|^2\right)^{\frac{1}{2}}$$

$$= q^{-1}|E||F| + 2q^d q^{\frac{d}{2}} (q^{-d}|E|)^{\frac{1}{2}} (q^{-d}|F|)^{\frac{1}{2}} = q^{-1}|E||F| + 2q^{\frac{d}{2}}|E|^{\frac{1}{2}}|F|^{\frac{1}{2}}.$$

□

From (5.3) in Lemma 5.1, we obtain the following.

COROLLARY 5.2. *Let $E, F \subset \mathbb{F}_q^d$ and $d \ge 2$ even. If $|E||F| \ge 16q^d$, then we have*

$$(|E||F| - \nu(0))^2 \ge \frac{|E|^2|F|^2}{36}.$$

PROOF. The inequality (5.3) in Lemma 5.1 implies that

$$|E||F| - \nu(0) \ge |E||F| - |\nu(0)| \ge (1-q^{-1})|E||F| - 2q^{\frac{d}{2}}|E|^{\frac{1}{2}}|F|^{\frac{1}{2}}.$$

Since $q \ge 3$ and $|E||F| \ge 16q^d$, we see that that

$$|E||F| - \nu(0) \ge \frac{2|E||F|}{3} - 2q^{\frac{d}{2}}|E|^{\frac{1}{2}}|F|^{\frac{1}{2}} \ge \frac{|E||F|}{6} \ge 0.$$

The statement follows immediately from this observation. □

We also need the following lemma.

LEMMA 5.3. *Suppose that $d \ge 2$ is even and $E, F \subset \mathbb{F}_q^d$. If $|E||F| \ge 16q^d$, then we have*

$$q^{3d} \left|\sum_{m\in S_0} \overline{\widehat{E}}(m)\widehat{F}(m)\right|^2 - \nu^2(0) \le q^{-1}|E|^2|F|^2.$$

PROOF. Recall from (5.2) in Lemma 5.1 that

$$\nu(0) = q^d G^d \sum_{m\in S_0} \overline{\widehat{E}}(m)\widehat{F}(m) + M(E,F),$$

where $M(E,F) = q^{-1}|E||F| - q^{d-1}G^d \sum_{m\in\mathbb{F}_q^d} \overline{\widehat{E}}(m)\widehat{F}(m)$. Since $\nu^2(0) = \nu(0)\overline{\nu(0)}$, we see that

$$\nu^2(0) = \left(q^d G^d \sum_{m\in S_0} \overline{\widehat{E}}(m)\widehat{F}(m) + M(E,F)\right)\left(q^d \overline{G^d} \sum_{m\in S_0} \widehat{E}(m)\overline{\widehat{F}}(m) + \overline{M(E,F)}\right).$$

Expanding the right-hand side and using the Gauss sum estimate, we see that

$$\nu^2(0) = q^{3d} \left|\sum_{m\in S_0} \overline{\widehat{E}}(m)\widehat{F}(m)\right|^2 + q^d G^d \overline{M(E,F)} \sum_{m\in S_0} \overline{\widehat{E}}(m)\widehat{F}(m)$$

$$+q^d\overline{G^d}M(E,F)\sum_{m\in S_0}\widehat{E}(m)\overline{\widehat{F}}(m)+|M(E,F)|^2.$$

Since $\nu^2(0)$ is a nonnegative integer and $|M(E,F)|^2\geq 0$, the equality above implies that

$$q^{3d}\left|\sum_{m\in S_0}\overline{\widehat{E}}(m)\widehat{F}(m)\right|^2-\nu^2(0)$$
$$\leq -q^dG^d\overline{M(E,F)}\sum_{m\in S_0}\overline{\widehat{E}}(m)\widehat{F}(m)-q^d\overline{G^d}M(E,F)\sum_{m\in S_0}\widehat{E}(m)\overline{\widehat{F}}(m)$$
$$\leq 2q^d|G^d||M(E,F)|\sum_{m\in S_0}|\widehat{E}(m)||\widehat{F}(m)|\leq 2q^{\frac{3d}{2}}|M(E,F)|\sum_{m\in\mathbb{F}_q^d}|\widehat{E}(m)||\widehat{F}(m)|.$$

Now, notice from the Cauchy-Schwarz inequality and the Plancherel theorem that

$$\sum_{m\in\mathbb{F}_q^d}|\widehat{E}(m)||\widehat{F}(m)|\leq q^{-d}|E|^{\frac{1}{2}}|F|^{\frac{1}{2}}.$$

Using this estimate and the definition of $M(E,F)$, it is easy to see that

$$|M(E,F)|\leq q^{-1}|E||F|+q^{d-1}|G^d|\sum_{m\in\mathbb{F}_q^d}|\widehat{E}(m)||\widehat{F}(m)|\leq q^{-1}|E||F|+q^{\frac{d}{2}-1}|E|^{\frac{1}{2}}|F|^{\frac{1}{2}}.$$

Putting all estimates above together gives that

$$q^{3d}\left|\sum_{m\in S_0}\overline{\widehat{E}}(m)\widehat{F}(m)\right|^2-\nu^2(0)\leq 2q^{\frac{d}{2}-1}|E|^{\frac{3}{2}}|F|^{\frac{3}{2}}+2q^{d-1}|E||F|.$$

A direct computation shows that if $|E||F|\geq 16q^d$, then

$$2q^{\frac{d}{2}-1}|E|^{\frac{3}{2}}|F|^{\frac{3}{2}}+2q^{d-1}|E||F|\leq 4q^{\frac{d}{2}-1}|E|^{\frac{3}{2}}|F|^{\frac{3}{2}}\leq q^{-1}|E|^2|F|^2.$$

This completes the proof. □

**5.2. Proof of the first part of Theorem 2.2.** Suppose that $d\geq 2$ is even and $E,F\subset\mathbb{F}_q^d$ with $|E||F|\geq 16q^d$. Applying Corollary 5.2 and Lemma 5.3 to the formula (5.1), we obtain that

$$|\Delta(E,F)|\geq\frac{|E|^2|F|^2/36}{2q^{-1}|E|^2|F|^2+q^{2d}|F|\left(\max_{r\in\mathbb{F}_q^*}\sum_{m\in S_r}|\widehat{E}(m)|^2\right)}. \tag{5.4}$$

As before, the Plancherel theorem yields

$$\max_{r\in\mathbb{F}_q^*}\sum_{m\in S_r}|\widehat{E}(m)|^2\leq\sum_{m\in\mathbb{F}_q^d}|\widehat{E}(m)|^2=q^{-d}|E|. \tag{5.5}$$

As before, the definition of the Fourier transform gives that for each $r\in\mathbb{F}_q$,

$$\sum_{m\in S_r}|\widehat{E}(m)|^2=q^{-d}\sum_{x,y\in E}\widehat{S_r}(x-y)=q^{-d}|E|\widehat{S_r}(0,\ldots,0)+q^{-d}\sum_{x,y\in E:x-y\neq(0,\ldots,0)}\widehat{S_r}(x-y).$$

Since $d\geq 2$ is even, applying the second part of Corollary 3.1 yields that

$$\max_{r\in\mathbb{F}_q^*}\sum_{m\in S_r}|\widehat{E}(m)|^2\leq 2q^{-d-1}|E|+2q^{-\frac{3d+1}{2}}|E|^2.$$

From this estimate and (5.5), it is clear that if $d \geq 2$ is even, then

$$\max_{r\in\mathbb{F}_q^*}\sum_{m\in S_r}|\widehat{E}(m)|^2 \leq \min\{q^{-d}|E|,\ 2q^{-d-1}|E|+2q^{-\frac{3d+1}{2}}|E|^2\}$$
$$\leq \begin{cases} 4q^{-d-1}|E| & \text{if } 1\leq |E|< q^{\frac{d-1}{2}}\\ 4q^{-\frac{3d+1}{2}}|E|^2 & \text{if } q^{\frac{d-1}{2}}\leq |E|<q^{\frac{d+1}{2}}\\ q^{-d}|E| & \text{if } q^{\frac{d+1}{2}}\leq |E|\leq q^d.\end{cases}$$

Combining this inequality with (5.4) and considering the dominant term in terms of $|E|$, we obtain from a direction calculation that

$$|\Delta(E,F)|\geq \begin{cases} \frac{1}{144}\min\left\{q,\ \frac{|E||F|}{2q^{d-1}}\right\} & \text{for } 1\leq |E|<q^{\frac{d-1}{2}}\\ \frac{1}{144}\min\left\{q,\ \frac{|F|}{2q^{\frac{d-1}{2}}}\right\} & \text{for } q^{\frac{d-1}{2}}\leq |E|<q^{\frac{d+1}{2}}\\ \frac{1}{144}\min\left\{q,\ \frac{2|E||F|}{q^d}\right\} & \text{for } q^{\frac{d+1}{2}}\leq |E|\leq q^d.\end{cases}$$

Since $|E||F|\geq 16q^d$, we conclude that

$$|\Delta(E,F)|\geq \begin{cases} \frac{q}{144} & \text{for } |E|<q^{\frac{d-1}{2}}\\ \frac{1}{144}\min\left\{q,\ \frac{|F|}{2q^{\frac{d-1}{2}}}\right\} & \text{for } q^{\frac{d-1}{2}}\leq |E|<q^{\frac{d+1}{2}}\\ \frac{1}{144}\min\left\{q,\ \frac{2|E||F|}{q^d}\right\} & \text{for } q^{\frac{d+1}{2}}\leq |E|\leq q^d.\end{cases}$$

Thus, the proof of the first part of Theorem 2.2 is complete.

**5.3. Proof of the second part of Theorem 2.2.** Suppose that $E,F\subset\mathbb{F}_q^2$ with $|E||F|\geq 16q^2$. Applying Lemma 3.6 to the inequality (5.4), we conclude that

$$|\Delta(E,F)|\geq \frac{|E|^2|F|^2}{36(2q^{-1}|E|^2|F|^2+\sqrt{3}q|E|^{\frac{3}{2}}|F|)}\geq \frac{1}{72}\min\left\{\frac{q}{2},\ \frac{|E|^{\frac{1}{2}}|F|}{\sqrt{3}q}\right\}.$$

**5.4. Proof of the third part of Theorem 2.2.** Let $E,F\subset\mathbb{F}_q^2$. In addition, assume that $-1\in\mathbb{F}_q$ is not a square number. Then $S_0=\{(0,0)\}$. Therefore, it follows from (3.6) in Lemma 3.5 that

$$\sum_{t\in\mathbb{F}_q}\nu^2(t)\leq q^{-1}|E|^2|F|^2+q^6\left|\overline{\widehat{E}}(0,\ldots,0)\widehat{F}(0,\ldots,0)\right|^2+q^4|F|\left(\max_{r\in\mathbb{F}_q^*}\sum_{m\in S_r}|\widehat{E}(m)|^2\right).$$

Since $\overline{\widehat{E}}(0,\ldots,0)=q^{-2}|E|$ and $\widehat{F}(0,\ldots,0)=q^{-2}|F|$, an application of Lemma 3.6 yields that

$$\sum_{t\in\mathbb{F}_q}\nu^2(t)\leq q^{-1}|E|^2|F|^2+q^{-2}|E|^2|F|^2+\sqrt{3}q|E|^{\frac{3}{2}}|F|.$$

Now, observe that $q^{-2}|E|^2|F|^2\leq q|E|^{\frac{3}{2}}|F|$ for $E,F\subset\mathbb{F}_q^2$. It therefore follows that

$$\sum_{t\in\mathbb{F}_q}\nu^2(t)\leq q^{-1}|E|^2|F|^2+(1+\sqrt{3})q|E|^{\frac{3}{2}}|F|.$$

By this inequality and (3.3) in Lemma 3.3, we see that

$$|\Delta(E,F)| \geq \frac{|E|^2|F|^2}{q^{-1}|E|^2|F|^2 + (1+\sqrt{3})q|E|^{\frac{3}{2}}|F|} \geq \min\left\{\frac{q}{2},\ \frac{|E|^{\frac{1}{2}}|F|}{2(\sqrt{3}+1)q}\right\}.$$

Thus, the proof is complete.

## 6. proof of Theorem 2.3

Let $d \geq 2$. Suppose that $E = A \times A \times \cdots \times A \subset \mathbb{F}_q^d$ is a product set and $F \subset \mathbb{F}_q^d$. Combining (3.3) in Lemma 3.3 with (3.5) in Lemma 3.5, we have

$$|\Delta(E,F)| \geq \frac{|E|^2|F|^2}{q^{-1}|E|^2|F|^2 + q^{2d}|F|\left(\max_{r\in\mathbb{F}_q}\sum_{m\in S_r}|\widehat{E}(m)|^2\right)}. \tag{6.1}$$

Let $E = \underline{E} \times A \subset \mathbb{F}_q^{d-1} \times \mathbb{F}_q$. Then we see from the definition of the Fourier transform that for $m = (\underline{m}, m_d) \in \mathbb{F}_q^{d-1} \times \mathbb{F}_q$,

$$\widehat{E}(m) = \widehat{\underline{E}\times A}(\underline{m}, m_d) = \widehat{\underline{E}}(\underline{m})\widehat{A}(m_d),$$

where $\widehat{\underline{E}}(\underline{m}) := q^{-(d-1)}\sum_{\underline{x}\in\mathbb{F}_q^{d-1}}\chi(-\underline{m}\cdot\underline{x})\underline{E}(\underline{x})$, and $\widehat{A}(m_d) := q^{-1}\sum_{s\in\mathbb{F}_q}\chi(-s\cdot m_d)A(s)$. Then, for each $r \in \mathbb{F}_q$, we can write

$$\sum_{m\in S_r}|\widehat{E}(m)|^2 = \sum_{\underline{m}\in\mathbb{F}_q^{d-1}}\left|\widehat{\underline{E}}(\underline{m})\right|^2\left(\sum_{m_d\in\mathbb{F}_q: m_d^2 = r-\|\underline{m}\|}|\widehat{A}(m_d)|^2\right).$$

Since $|\widehat{A}(m_d)| \leq |\widehat{A}(0)| = q^{-1}|A|$ for all $m_d \in \mathbb{F}_q$, and $|\{m_d \in \mathbb{F}_q : m_d^2 = r - \|\underline{m}\|\}| \leq 2$ for each $r \in \mathbb{F}_q, \underline{m} \in \mathbb{F}_q^{d-1}$, we see that

$$\sum_{m\in S_r}|\widehat{E}(m)|^2 \leq 2q^{-2}|A|^2\sum_{\underline{m}\in\mathbb{F}_q^{d-1}}\left|\widehat{\underline{E}}(\underline{m})\right|^2 = 2q^{-d-1}|A|^2|\underline{E}|,$$

where the last equality follows from the Plancherel theorem in dimension $(d-1)$. Since $E = A \times \cdots \times A = \underline{E} \times A$ is a product set, it is clear that $|A|^2|\underline{E}| = |E|^{1+\frac{1}{d}}$. Hence, we obtain that

$$\max_{r\in\mathbb{F}_q}\sum_{m\in S_r}|\widehat{E}(m)|^2 \leq 2q^{-d-1}|E|^{1+\frac{1}{d}}.$$

Applying this inequality to (6.1) gives

$$|\Delta(E,F)| \geq \frac{|E|^2|F|^2}{q^{-1}|E|^2|F|^2 + 2q^{d-1}|E|^{1+\frac{1}{d}}|F|} \geq \min\left\{\frac{q}{2}, \frac{|E|^{1-\frac{1}{d}}|F|}{4q^{d-1}}\right\}.$$

This completes the proof.

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