The generalized Erdös-Falconer distance problems in vector spaces over finite fields

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Abstract. In this paper we study the generalized Erdös-Falconer distance problems in the finite field setting. The generalized distances are defined in terms of polynomials, and various formulas for sizes of distance sets are obtained. In particular, we develop a simple formula for estimating the cardinality of distance sets determined by diagonal polynomials. As a result, we generalize the spherical distance problems due to Iosevich and Rudnev [13] and the cubic distance problems due to Iosevich and Koh [12]. Moreover, our results are higher-dimensional version of Vu's work [24] on two dimensions. In addition, we set up and study the generalized pinned distance problems in finite fields. We give a generalization of the work by the authors [2] who studied the pinned distance problems related to spherical distances. Discrete Fourier analysis and exponential sum estimates play an important role in our proof.

Contents

1. Introduction

The Erdős distance problem, in a generalized sense, is a question of how many distances are determined by a set of points. This problem might be the most well known problem in discrete geometry. One may consider discrete, continuous and finite field formulations of this question. Given finite subsets E, F of $\mathbf{R^d}$, $d \geq 2$, the distance set determined by the sets E, F is defined by $\Delta(E,F) = \{|x-y| : x \in E, y \in F\}$, where $|x| = \sqrt{x_1^2 + \cdots + x_d^2}$. In the case when $E = F$, Erdős [7] asked us to determine the smallest possible size of $\Delta(E,E)$ in terms of the size of E. This problem is called the Erdős distance problem and it was conjectured that

$$|\Delta(E,E)| \gtrapprox |E|^{2/d},$$

where $|\cdot|$ denotes the cardinality of the finite set and $|E|$ is the controlling parameter. Here we recall that $A \gtrapprox B$, with the controlling parameter N, means that for each $\varepsilon > 0$ there exists $C_\varepsilon > 0$

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such that $C_\varepsilon N^\varepsilon A \geq B$ where $C_\varepsilon > 0$ is independent of the controlling parameter N. Taking E as a piece of the integer lattice shows that one can not in general get the better exponent than $2/d$ for the conjecture. In dimension two, the conjecture was solved by Guth and Katz [9]. For the best known results in dimension $d \geq 3$ see [21] and [22]. These results are a culmination of efforts going back to the paper by Erdős [7].

On the other hand, one can also study the continuous analog of the Erdős distance problem, called the Falconer distance problem. This problem is to determine the Hausdorff dimension of compact sets such that the Lebesgue measure of the distance sets is positive. Let $E \subset \mathbf{R^d}, d \geq 2$, be a compact set. The Falconer distance conjecture says that if $\dim(E) > d/2$, then $|\Delta(E,E)| > 0$, where $\dim(E)$ denotes the Hausdorff dimension of the set E, and $|\Delta(E,E)|$ denotes one dimensional Lebesgue measure of the distance set $\Delta(E,E) = \{|x-y| : x, y \in E\}$. Using the Fourier transform method, Falconer [8] proved that if $\dim(E) > (d+1)/2$, then $|\Delta(E,E)| > 0$. This result was generalized by Mattila [18] who showed that

$$\text{if} \quad \dim(E) + \dim(F) > d+1, \text{ then } |\Delta(E,F)| > 0,$$

where E, F are compact subsets of $\mathbf{R^d}$ and $\Delta(E,F) = \{|x-y| \in \mathbf{R} : x \in E, y \in F\}$. In particular, he made a remarkable observation that the Falconer distance problem is closely related to estimating the upper bound of the spherical means of Fourier transforms of measures. Using Mattila's method, Wolff [26] obtained the best known result on the Falconer distance problem in dimension two. He proved that if $\dim(E) > 4/3$, then $|\Delta(E,E)| > 0$. The best known results for higher dimensions are due to Erdoğan [6]. Applying Mattila's method and the weighted version of Tao's bilinear extension theorem [23], he proved that if $\dim(E) > d/2 + 1/3$, then $|\Delta(E,E)| > 0$, where $d \geq 2$ is the dimension. However, the Falconer distance problem is still open for all dimensions $d \geq 2$. As a variation of the Falconer distance problem, Peres and Schlag [19] studied the pinned distance problems and showed that the Falconer result can be sharpened. More precisely, they proved that if $E \subset \mathbf{R^d}$ and $\dim(E) > (d+1)/2$, then $|\Delta(E,y)| > 0$ for almost every $y \in E$, where the pinned distance set $\Delta(E,y)$ is given by

$$\Delta(E,y) = \{|x-y| : x \in E\}.$$

In recent years the Erdős-Falconer distance problem has been also studied in the finite field setting. Let $\mathbb{F}_q$ be a finite field with q elements. We denote by $\mathbb{F}_q^d, d \geq 2$, the d-dimensional vector space over the finite field $\mathbb{F}_q$. Given a polynomial $P(x) \in \mathbb{F}_q[x_1, \ldots, x_d]$ and $E, F \subset \mathbb{F}_q^d$, one may define a generalized distance set $\Delta_P(E,F)$ by the set

$$\Delta_P(E,F) = \{P(x-y) \in \mathbb{F}_q : x \in E, y \in F\}. \tag{1.1}$$

Throughout the paper we assume that the degree of any polynomial is greater than equal to two. In the case when $E = F$ and $P(x) = x_1^2 + x_2^2$, Bourgain, Katz and Tao [1] first obtained the following nontrivial result on the Erdős distance problem in the finite field setting: if q is prime with $q \equiv 3 \pmod 4$ and $E \subset \mathbb{F}_q^2$ with $|E| = q^\delta$ for some $0 < \delta < 2$, then there exists $\varepsilon = \varepsilon(\delta) > 0$ such that

$$|\Delta_P(E,E)| \gtrsim |E|^{\frac{1}{2}+\varepsilon}, \tag{1.2}$$

where we recall that if A, B are positive numbers, then $A \gtrsim B$ means that there exists $C > 0$ independent of q, the cardinality of the underlying finite field $\mathbb{F}_q$ such that $CA \geq B$, and $A \sim B$ means $A \gtrsim B$ and $B \gtrsim A$. However, if there exists $i \in \mathbb{F}_q$ with $i^2 = -1$, or the field $\mathbb{F}_q$ is not the prime field, then the inequality (1.2) can not be true in general. For example, if we take $E = \{(s, is) \in \mathbb{F}_q^2 : s \in \mathbb{F}_q\}$, then $|E| = q$ but $|\Delta_P(E,E)| = |\{0\}| = 1$. Moreover, if $q = p^2$ with p prime, and $E = \mathbb{F}_p^2$, then $|E| = p^2 = q$ but $|\Delta_P(E,E)| = p = \sqrt{q}$. In view of these examples, Iosevich and Rudnev [13] replaced the question on the Erdős distance problems by the following Falconer distance problem in the finite field setting: how large a set $E \subset \mathbb{F}_q^d$ is needed to obtain a positive proportion of all distances. They first showed that if $|E| \geq 2q^{(d+1)/2}$ then one can obtain all distances; that is $|\Delta_P(E,E)| = q$ where $P(x) = x_1^2 + \cdots + x_d^2$. In addition, they conjectured that $|E| \gtrsim q^{\frac{d}{2}}$ implies that $|\Delta_P(E,E)| \gtrsim q$. In the case when $P(x) = x_1^k + \cdots + x_d^k, k \geq 2$, more general conjecture was given by Iosevich and Koh [12]. However, it turned out that in the case $k = 2$ if one wants to obtain all distances, then arithmetic examples constructed by authors in [10] show that the exponent $(d+1)/2$ is sharp in odd dimensions. The problems in even dimensions are still open. Moreover if one wants to obtain a positive proportion of all distances, then the exponent $(d+1)/2$ was recently improved in two dimensions by the authors in [2] who proved that if $E \subset \mathbb{F}_q^2$ with $|E| \gtrsim q^{4/3}$, then $|\Delta_P(E,E)| \gtrsim q$ where $P(x) = x_1^2 + x_2^2$. This result was generalized by Koh and Shen [16] in the sense that if $E, F \subset \mathbb{F}_q^2$ and $|E||F| \gtrsim q^{8/3}$, then $|\Delta_P(E,F)| = |\{P(x-y) \in \mathbb{F}_q : x \in E, y \in F\}| \gtrsim q$.

In this paper, we shall study the Erdős-Falconer distance problems for finite fields, associated with the generalized distance set defined as in (1.1). This problem can be considered as a generalization of the spherical distance problems and the cubic distance problems which were studied by Iosevich and Rudnev in [13] and Iosevich and Koh in [12] respectively. The generalized Erdős distance problem was first introduced by Vu [24], mainly studying the size of the distance sets, generated by nondegenerate polynomials $P(x) \in \mathbb{F}_q[x_1, x_2]$. Using the spectral graph theory, he proved that if $P(x) \in \mathbb{F}_q[x_1, x_2]$ is a nondegenerate polynomial and $E \subset \mathbb{F}_q^2$ with $|E| \gtrsim q$, then we have

$$|\Delta_P(E,E)| \gtrsim \min\left(q, |E|q^{-\frac{1}{2}}\right) \tag{1.3}$$

where a polynomial $P(x) \in \mathbb{F}_q[x_1, x_2]$ is called a nondegenerate polynomial if it is not of the form $G(L(x_1, x_2))$ where G is an one-variable polynomial and L is a linear form in x_1, x_2. In order to obtain the inequality (1.3), the assumption $|E| \gtrsim q$ is necessary in general setting, which is clear from the following example: if $P(x) = x_1^2 - x_2^2$ and $E = \{(t,t) \in \mathbb{F}_q^2 : t \in \mathbb{F}_q\}$ is the line, then we see that $|E| = q$ and $|\Delta_P(E,E)| = |\{0\}| = 1$ and so the inequality (1.3) can not be true. Using the Fourier analysis method, Hart, Li, and Shen [11] showed that $P(x) - b \in \mathbb{F}_q[x_1, x_2]$ does not have any linear factor for all $b \in \mathbb{F}_q$ if and only if the following inequality holds:

$$|\Delta_P(E,F)| \gtrsim \min\left(q, \sqrt{|E||F|}q^{-\frac{1}{2}}\right) \quad \text{for all } E, F \subset \mathbb{F}_q^2. \tag{1.4}$$

In the finite field setting, results on the Erdős distance problem implies results on the Falconer distance problem. For example, the inequality (1.4) implies that if $E, F \subset \mathbb{F}_q^2$ with $|E||F| \gtrsim q^3$, then $\Delta_P(E,F)$ contains a positive proportion of all possible distances; that is $|\Delta_P(E,F)| \gtrsim q$.

The purpose of this paper is to develop the two-dimensional work by Vu [24] to higher dimensions. In terms of the Fourier decay on varieties generated by general polynomials, we classify the size of distance sets. In particular, we investigate the size of the generalized Erdős-Falconer distance sets related to diagonal polynomials, that are of the form

$$P(x) = \sum_{j=1}^{d} a_j x_j^{k_j} \in \mathbb{F}_q[x_1, \ldots, x_d]$$

where $a_j \neq 0$ and $k_j \geq 2$ for all $i = 1, \ldots, d$. The polynomial $P(x) = \sum_{j=1}^d x_j^2$ is related to the spherical distance problem. In this case, the Erdős-Falconer distance problems were well studied by Iosevich and Rudnev [13]. On the other hand, Iosevich and Koh [12] studied the cubic distance problems associated with the polynomial $P(x) = \sum_{j=1}^d x_j^3$. In addition, Vu's theorem (1.3) gives us some results on the Erdős-Falconer distance problems in dimension two related to the polynomial $P(x) = a_1x_1^{k_1} + a_2x_2^{k_2}$. As we shall see, our results will recover and extend the aforementioned authors' work. Moreover, we address here that the arguments in the work mentioned before can not be directly applied to our cases. In part, it is not easy to obtain a sharp Fourier decay estimate for the varieties associated with the generalized polynomials considered in this paper. We will get over the difficulties by considering the sets as product sets (see section 4 for details). In addition, we also study the generalized pinned distance problems in the finite field setting in which our result sharpens and generalizes Vu's result (1.3). The authors in [2] considered the following pinned distance set:

$$\Delta_P(E, y) = \{P(x-y) \in \mathbb{F}_q : x \in E\}$$

where $E \subset \mathbb{F}_q^d, y \in \mathbb{F}_q^d$ and $P(x) = x_1^2 + \cdots + x_d^2$. They proved that if $E \subset \mathbb{F}_q^d, d \geq 2$, and $|E| \geq q^{\frac{d+1}{2}}$, then there exists $E' \subset E$ with $|E'| \sim |E|$ such that

$$|\Delta_P(E, y)| > \frac{q}{2} \quad \text{for all } y \in E'. \tag{1.5}$$

One of the most important ingredients in the proof is that this specific polynomial $P(x)$ has the following crucial property: namely, for $x, x', y \in \mathbb{F}_q^d$,

$$P(x-y) - P(x'-y) = (P(x) - 2y \cdot x) - (P(x') - 2y \cdot x'). \tag{1.6}$$

However, if the polynomial $P(x)$ is replaced by a general polynomial in $\mathbb{F}_q[x_1, \ldots, x_d]$, then the equality (1.6) can not be in general obtained. Investigating the Fourier decay on the variety generated by a polynomial enables us to prove the above pinned distance result (1.5) for general polynomials $P(x)$. For instance, our result implies that facts such as the above result (1.5) can be obtained if the polynomial P is a diagonal polynomial with all exponents equal.

2. Discrete Fourier analysis and exponential sums

In order to prove our main results on the generalized Erdős-Falconer distance problems, the discrete Fourier analysis shall be used as the principle tool. In this section, we review the discrete Fourier analysis machinery for finite fields, and collect some well known facts on classical exponential sums.

2.1. Finite Fourier analysis. Let $\mathbb{F}_q^d, d \geq 2$, be a d-dimensional vector space over the finite field $\mathbb{F}_q$ with q element. We shall work on the vector space $\mathbb{F}_q^d$, and throughout the paper, we shall assume that the characteristic of the finite field $\mathbb{F}_q$ is sufficiently large so that some minor technical problems can be overcome. Now, let us review the definition of the canonical additive character of $\mathbb{F}_q$. Let $q = p^s$ with p prime. Recall that the trace function $Tr : \mathbb{F}_q \to \mathbb{F}_p$ is defined by

$$Tr(c) = c + c^p + \cdots + c^{p^{s-1}} \quad \text{for } c \in \mathbb{F}_q.$$

We identify $\mathbb{F}_p$ with $\mathbb{Z}/(p)$. Then the function χ defined by $\chi(a) = e^{2\pi i Tr(a)/p}$ for all $a \in \mathbb{F}_q$ is called the canonical additive character of $\mathbb{F}_q$. For example, if q is prime, then $\chi(s) = e^{2\pi i s/q}$. Throughout the paper we denote by χ the canonical additive character of $\mathbb{F}_q$. Let $f : \mathbb{F}_q^d \to \mathbb{C}$ be a complex valued function on $\mathbb{F}_q^d$. Then, the Fourier transform of the function f is defined by

$$\widehat{f}(m) = \frac{1}{q^d} \sum_{x \in \mathbb{F}_q^d} f(x)\chi(-x \cdot m) \quad \text{for } m \in \mathbb{F}_q^d. \tag{2.1}$$

We also recall in this setting that the Fourier inversion theorem says that

$$f(x) = \sum_{m \in \mathbb{F}_q^d} \chi(x \cdot m)\widehat{f}(m). \tag{2.2}$$

Using the orthogonality relation of the canonical additive character χ; that is $\sum_{x\in\mathbb{F}_q^d}\chi(x\cdot m)=0$ for $m\neq(0,\ldots,0)$ and $\sum_{x\in\mathbb{F}_q^d}\chi(x\cdot m)=q^d$ for $m=(0,\ldots,0)$, we obtain the following Plancherel theorem:

$$\sum_{m\in\mathbb{F}_q^d}|\widehat{f}(m)|^2=\frac{1}{q^d}\sum_{x\in\mathbb{F}_q^d}|f(x)|^2.$$

For example, if f is a characteristic function on the subset E of $\mathbb{F}_q^d$, then we see

$$\sum_{m\in\mathbb{F}_q^d}|\widehat{E}(m)|^2=\frac{|E|}{q^d}. \tag{2.3}$$

Here, and throughout the paper, we identify the set $E\subset\mathbb{F}_q^d$ with the characteristic function on the set E, and we denotes by $|E|$ the cardinality of the set $E\subset\mathbb{F}_q^d$.

2.2. Exponential sums. Using the discrete Fourier analysis, we shall make an effort to reduce the generalized Erdős-Falconer distance problems to estimating classical exponential sums. Some of our formulas for the distance problems can be directly applied via recent well known exponential sum estimates. For example, the following lemma is well known and it was obtained by applying cohomological arguments (see Example 4.4.19 in [3]).

Lemma 2.1. *Let $P(x)=\sum_{j=1}^{d}a_jx_j^k\in\mathbb{F}_q[x_1,\ldots,x_d]$ with $k\geq 2, a_j\neq 0$ for all $j=1,\ldots,d$, and $V_t=\{x\in\mathbb{F}_q^d : P(x)=t\}$. In addition, assume that the characteristic of $\mathbb{F}_q$ is sufficiently large so that it does not divide k. Then,*

$$|\widehat{V_t}(m)|=\frac{1}{q^d}\left|\sum_{x\in V_t}\chi(-x\cdot m)\right|\lesssim q^{-\frac{d+1}{2}}\quad \text{for all } m\in\mathbb{F}_q^d\setminus\{(0,\ldots,0)\}, t\in\mathbb{F}_q\setminus\{0\},$$

and

$$|\widehat{V_0}(m)|\lesssim q^{-\frac{d}{2}}\quad \text{for all } m\in\mathbb{F}_q^d\setminus\{(0,\ldots,0)\}.$$

However, some theorems obtained by cohomological arguments contain abstract assumptions, and it can be often hard to apply them in practice. In order to overcome this problem, we shall also develop an alternative formula which is closely related to more simple exponential sums. As we shall see, such a simple formula can be obtained by viewing the distance problem in d dimensions as the distance problem for product sets in $(d+1)-$dimensional vector spaces. As a typical application of our simple distance formula, we shall obtain the results on the Falconer distance problems related to arbitrary diagonal polynomials, which take the following forms: $P(x)=\sum_{j=1}^{d}a_jx_j^{k_j}$ for $k_j\geq 2, a_j\neq 0$ for all j. It is shown that such results can be obtained by applying the following well known Weil's theorem. For a nice proof of Weil's theorem, we refer readers to Theorem 5.38 in [17].

Theorem 2.2. *Let $f\in\mathbb{F}_q[s]$ be of degree $k\geq 1$ with $gcd(k,q)=1$. Then, we have*

$$\left|\sum_{s\in\mathbb{F}_q}\chi(f(s))\right|\leq(k-1)q^{\frac{1}{2}},$$

where χ denotes an additive character of $\mathbb{F}_q$.

We now collect well known facts which play a crucial role in the proof of our main results. First, we introduce the cardinality of varieties related to arbitrary diagonal polynomials. The following theorem is due to Weil [25]. See also Theorem 3.35 in [3] or Theorem 6.34 in [17].

THEOREM 2.3. *Let* $P(x) = \sum_{j=1}^{d} a_j x_j^{k_j}$ *with* $a_j \neq 0, k_j \geq 1$ *for all* $j = 1, \ldots, d$. *For every* $t \in \mathbb{F}_q \setminus \{0\}$, *we have*

$$|V_t| \sim q^{d-1}.$$

The following lemma is known as the Schwartz-Zippel lemma (see [27] and [20]). A nice proof is also given in Theorem 6.13 in [17].

LEMMA 2.4. *Let* $P(x) \in \mathbb{F}_q[x_1, \ldots, x_d]$ *be a nonzero polynomial with degree* k. *Then, we have*

$$|V_0| \leq k q^{d-1}.$$

We also need the following theorem which is a corollary of Theorem 5.1.1 in [14].

THEOREM 2.5. *Let* $P(x) \in \mathbb{F}_q[x_1, x_2]$ *be a nondegenerate polynomial of degree* $k \geq 2$. *Then there is a set* $T \subset \mathbb{F}_q$ *with* $0 \leq |T| \leq (k-1)$, *such that for every* $m \in \mathbb{F}_q^2 \setminus \{(0,0)\}, t \notin T$,

$$|\widehat{V_t}(m)| = \frac{1}{q^2} \left| \sum_{x \in V_t} \chi(-x \cdot m) \right| \lesssim q^{-\frac{3}{2}},$$

where $V_t = \{x \in \mathbb{F}_q^2 : P(x) = t\}$ *for* $t \in \mathbb{F}_q$.

REMARK 2.6. In Theorem 2.5, it is clear that if $t \in T$, then

$$|\widehat{V_t}(m)| \lesssim q^{-1} \quad \text{for all } m \in \mathbb{F}_q^2. \tag{2.4}$$

This follows immediately from the Schwartz-Zippel lemma and the simple observation that $|\widehat{V_t}(m)| \leq q^{-2}|V_t|$.

3. Distance formulas based on the Fourier decays

Following the similar skills due to Iosevich and Rudnev [13], we shall obtain the generalized distance formulas. As an application of the formulas, we will obtain results on the generalized Erdős-Falconer distance problems associated with specific diagonal polynomials $P(x) = \sum_{j=1}^{d} a_j x_j^{k}$. Let $P(x) \in \mathbb{F}_q[x_1, \ldots, x_d]$ be a polynomial with degree ≥ 2. Given sets $E, F \subset \mathbb{F}_q^d$, recall that a generalized pair-wise distance set $\Delta_P(E,F)$ is given by the set

$$\Delta_P(E,F) = \{P(x-y) \in \mathbb{F}_q : x \in E, y \in F\}.$$

For the Erdős distance problems, we aim to find the lower bound of $|\Delta_P(E,F)|$ in terms of $|E|, |F|$. For the Falconer distance problems, our goal is to determine an optimal exponent $s_0 > 0$ such that if $|E||F| \gtrsim q^{s_0}$, then $|\Delta_P(E,F)| \gtrsim q$. In this general setting, the main difficulty on these problems is that we do not know the explicit form of the polynomial $P(x) \in \mathbb{F}_q[x_1, \ldots, x_d]$, generating generalized distances. Thus, we first try to find some conditions on the variety $V_t = \{x \in \mathbb{F}_q^d : P(x) = t\}$ for $t \in \mathbb{F}_q$ such that some results can be obtained for the distance problems. In view of this idea, we have the following distance formula.

THEOREM 3.1. *Let* $E, F \subset \mathbb{F}_q^d$ *and* $P(x) \in \mathbb{F}_q[x_1, \ldots, x_d]$. *For each* $t \in \mathbb{F}_q$, *we let*

$$V_t = \{x \in \mathbb{F}_q^d : P(x) - t = 0\}. \tag{3.1}$$

Suppose that there is a set $T \subset \mathbb{F}_q$ *such that* $|V_t| \sim q^{d-1}$ *for all* $t \in \mathbb{F}_q \setminus T$ *and*

$$\left|\widehat{V_t}(m)\right| \lesssim q^{-\frac{d+1}{2}} \quad \textit{for all} \quad t \notin T, m \in \mathbb{F}_q^d \setminus \{(0, \ldots, 0)\}. \tag{3.2}$$

Then, if $|E||F| \gtrsim q^{d+1}$, *we have*

$$|\Delta_P(E,F)| \geq q - |T|.$$

Proof. Consider the counting function ν on $\mathbb{F}_q$ given by

$$\nu(t) = |\{(x,y) \in E \times F : P(x-y) = t\}|\,.$$

It suffices to show that $\nu(t) \neq 0$ for every $t \in \mathbb{F}_q \setminus T$. Fix $t \notin T$. Applying the Fourier inversion theorem (2.2) to $V_t(x-y)$ and using the definition of the Fourier transform (2.1), we have

$$\nu(t) = \sum_{x\in E, y\in F} V_t(x-y) = q^{2d} \sum_{m \in \mathbb{F}_q^d} \overline{\widehat{E}}(m)\widehat{F}(m)\widehat{V_t}(m),$$

where we also used the simple fact that $\overline{\widehat{E}}(m) = q^{-d}\sum_{x\in E}\chi(x\cdot m)$. Write $\nu(t)$ by

$$\begin{aligned} \nu(t) =& q^{2d}\overline{\widehat{E}}(0,\dots,0)\widehat{F}(0,\dots,0)\widehat{V_t}(0,\dots,0) + q^{2d}\sum_{m\in\mathbb{F}_q^d\setminus\{(0,\dots,0)\}} \overline{\widehat{E}}(m)\widehat{F}(m)\widehat{V_t}(m) \\ =& \mathrm{I} + \mathrm{II}. \end{aligned} \tag{3.3}$$

From the definition of the Fourier transform, we see

$$0 < \mathrm{I} = \frac{1}{q^d}|E||F||V_t|. \tag{3.4}$$

On the other hand, the estimate (3.2) and the Cauchy-Schwarz inequality yield

$$|\mathrm{II}| \lesssim q^{2d} q^{-\frac{d+1}{2}} \left(\sum_m \left|\overline{\widehat{E}}(m)\right|^2\right)^{\frac12} \left(\sum_m \left|\widehat{F}(m)\right|^2\right)^{\frac12}.$$

Applying the Plancherel theorem (2.3), we obtain

$$|\mathrm{II}| \lesssim q^{\frac{d-1}{2}} |E|^{\frac12}|F|^{\frac12}. \tag{3.5}$$

Since $|V_t| \sim q^{d-1}$ for each $t \in \mathbb{F}_q \setminus T$, comparing (3.4) with (3.5) gives the complete proof. $\square$

As a generalized version of spherical distance problems in [**13**] and cubic distance problems in [**12**], we have the following corollary.

Corollary 3.2. *Let $P(x) = \sum_{j=1}^d a_j x_j^k \in \mathbb{F}_q[x_1,\dots,x_d]$ for $k \geq 2$ integer and $a_j \neq 0$. Suppose that the characteristic of $\mathbb{F}_q$ is sufficiently large. If $|E||F| \gtrsim q^{d+1}$ for $E, F \subset \mathbb{F}_q^d$, then $|\Delta_P(E,F)| \geq q-1$.*

Proof. The statement in Corollary 3.2 follows immediately from Theorem 3.1 along with Lemma 2.1 and Theorem 2.3. $\square$

Under the assumptions in Corollary 3.2, we do not know whether the distance set $\Delta_P(E,F)$ contains zero or not. However, if $E \cap F \neq \emptyset$, then obviously $0 \in \Delta_P(E,F)$. In this case, the distance set contains all possible distances.

Theorem 3.1 may provide us with an exact size of distance set $\Delta_P(E,F)$ and it may be a useful theorem for the Falconer distance problems for finite fields. However, if $|E||F|$ is much smaller than q^{d+1}, then Theorem 3.1 does not give any information about the size of the distance set $\Delta_P(E,F)$. Now, we introduce another generalized distance formula which is useful for the Erdős distance problems in the finite field setting.

Theorem 3.3. *Let $E, F \subset \mathbb{F}_q^d$ and $P(x) \in \mathbb{F}_q[x_1,\dots,x_d]$. For each $t \in \mathbb{F}_q$, the variety V_t is defined as in (3.1). Suppose that there exists a set $A \subset \mathbb{F}_q$ with $|A| \sim 1$ such that*

$$|\widehat{V_t}(m)| \lesssim q^{-\frac{d+1}{2}} \quad \text{for all } t \notin A, m \in \mathbb{F}_q^d \setminus \{(0,\dots,0)\} \tag{3.6}$$

and

$$|\widehat{V_t}(m)| \lesssim q^{-\frac{d}{2}} \quad \textit{for all } t \in A, m \in \mathbb{F}_q^d \setminus \{(0,\ldots,0)\}. \tag{3.7}$$

If $|E||F| \gtrsim q^d$, *then we have*

$$|\Delta_P(E,F)| \gtrsim \min\left(q, q^{-\frac{(d-1)}{2}}\sqrt{|E||F|}\right).$$

Proof. From (3.3) and (3.4), we see that for every $t \in \mathbb{F}_q$,

$$\nu(t) = |\{(x,y) \in E \times F : P(x-y) = t\}|$$

$$= \frac{1}{q^d}|E||F||V_t| + q^{2d} \sum_{m \in \mathbb{F}_q^d \setminus \{(0,\ldots,0)\}} \overline{\widehat{E}}(m)\widehat{F}(m)\widehat{V_t}(m).$$

$$\lesssim \frac{|E||F|}{q} + q^{2d}\left(\max_{m \neq (0,\ldots,0)} |\widehat{V_t}(m)|\right) \sum_{m \in \mathbb{F}_q^d} |\overline{\widehat{E}}(m)||\widehat{F}(m)|,$$

where we also used the Schwartz-Zippel lemma (Theorem 2.4). From the Cauchy-Schwarz inequality and the Plancherel theorem (2.3), we therefore see that for every $t \in \mathbb{F}_q$,

$$\nu(t) \lesssim \frac{|E||F|}{q} + q^d\sqrt{|E||F|}\left(\max_{m \neq (0,\ldots,0)} |\widehat{V_t}(m)|\right).$$

From our hypotheses (3.6), (3.7), it follows that

$$\nu(t) \lesssim \frac{|E||F|}{q} + q^{\frac{d-1}{2}}\sqrt{|E||F|} \quad \text{if } t \notin A$$

and

$$\nu(t) \lesssim \frac{|E||F|}{q} + q^{\frac{d}{2}}\sqrt{|E||F|} \quad \text{if } t \in A.$$

By these inequalities and the definition of the counting function $\nu(t)$, we see that

$$|E||F| = \sum_{t \in \Delta_P(E,F)} \nu(t) = \sum_{t \in A \cap \Delta_P(E,F)} \nu(t) + \sum_{t \in (\mathbb{F}_q \setminus A) \cap \Delta_P(E,F)} \nu(t)$$

$$\lesssim \frac{|E||F|}{q} + q^{\frac{d}{2}}\sqrt{|E||F|} + \left(\frac{|E||F|}{q} + q^{\frac{d-1}{2}}\sqrt{|E||F|}\right)|\Delta_P(E,F)|,$$

where we used the fact that $|A| \sim 1$. Note that if $|E||F| \geq Cq^d$ for some $C > 0$ sufficiently large, then $|E||F| \sim |E||F| + \frac{|E||F|}{q} + q^{\frac{d}{2}}\sqrt{|E||F|}$. From this fact and the above estimate, we conclude that if $|E||F| \geq Cq^d$ for some $C > 0$ sufficiently large, then

$$|\Delta_P(E,F)| \gtrsim \frac{|E||F|}{\frac{|E||F|}{q} + q^{\frac{d-1}{2}}\sqrt{|E||F|}}$$

which completes the proof. □

Remark 3.4. From the proof of Theorem 3.3, it is clear that if A is an empty set, then we can drop the assumption that $|E||F| \geq Cq^d$ for some $C > 0$ sufficiently large. As an example showing that A can be an empty set, the authors in [15] showed that if the dimension $d \geq 3$ is odd and $P(x) = \sum_{j=1}^d a_j x_j^2$ with $a_j \neq 0$, then $|\widehat{V_t}(m)| \lesssim q^{-(d+1)/2}$ for all $m \neq (0,\ldots,0), t \in \mathbb{F}_q$.

Combining Theorem 3.3 with Lemma 2.1, the following corollary immediately follows.

Corollary 3.5. *Let $P(x) = \sum_{j=1}^d a_j x_j^k \in \mathbb{F}_q[x_1, \ldots, x_d]$ for $k \geq 2$ integer and $a_j \neq 0$. Assume that the characteristic of $\mathbb{F}_q$ is sufficiently large. If $E, F \subset \mathbb{F}_q^d$ with $|E||F| \gtrsim q^d$, then we have*

$$|\Delta_P(E,F)| \gtrsim \min\left(q, q^{-\frac{(d-1)}{2}}\sqrt{|E||F|}\right).$$

As pointed out in Remark 3.4, if $k = 2$ and d is odd, then the conclusion in Corollary 3.5 holds without the assumption that $|E||F| \gtrsim q^d$.

4. Simple formula for generalized Falconer distance problems

In the previous section, we have seen that the distance problems are closely related to decays of the Fourier transforms on varieties. In order to apply Theorem 3.1 or Theorem 3.3, we must estimate the Fourier decay of the variety $V_t = \{x \in \mathbb{F}_q^d : P(x) = t\}$. In general, it is not easy to estimate the Fourier transform of V_t. To do this, we need to show the following exponential sum estimate holds: for $m \in \mathbb{F}_q^d \setminus \{(0, \ldots, 0)\}$,

$$|\widehat{V_t}(m)| = q^{-d}\left|\sum_{x \in V_t} \chi(-x \cdot m)\right| = q^{-d-1}\left|\sum_{(x,s) \in \mathbb{F}_q^{d+1}} \chi(sP(x) - m \cdot x - st)\right| \lesssim q^{-\frac{d+1}{2}},$$

where the second equality follows from the orthogonality relation of the canonical additive character χ. In other words, we must show that for $m \neq (0, \ldots, 0)$,

$$\left|\sum_{(x,s) \in \mathbb{F}_q^{d+1}} \chi(sP(x) - m \cdot x - st)\right| \lesssim q^{\frac{d+1}{2}}. \tag{4.1}$$

Can we find a more useful, easier formula for distance problems than the formulas given in Theorem 3.1 or Theorem 3.3 ? If we are just interested in getting the positive proportion of all distances, then the answer is yes. We do not need to estimate the size of V_t and we just need to estimate more simple exponential sums. We have the following simple formula.

Theorem 4.1. *Let $P(x) \in \mathbb{F}_q[x_1, \ldots, x_d]$ be a polynomial with degree ≥ 2. Given $E, F \subset \mathbb{F}_q^d$, define the distance set*

$$\Delta_P(E,F) = \{P(x-y) \in \mathbb{F}_q : x \in E, y \in F\}.$$

Suppose that the following estimate holds: for every $m \in \mathbb{F}_q^d$ and $s \neq 0$,

$$\left|\sum_{x \in \mathbb{F}_q^d} \chi(sP(x) + m \cdot x)\right| \lesssim q^{\frac{d}{2}}. \tag{4.2}$$

Then, if $|E||F| \gtrsim q^{d+1}$, then $|\Delta_P(E,F)| \gtrsim q$.

Notice that the estimate (4.2) is weaker than the estimate (4.1). We shall see that Theorem 4.1 can be obtained by studying the distance problem related to the generalized paraboloid in $\mathbb{F}_q^{d+1}$. The details and the proof of Theorem 4.1 will be given in the next subsections. Using Theorem 4.1, we have the following corollary.

Corollary 4.2. *Let $P(x) = \sum_{j=1}^d a_j x_j^{k_j}$ for $k_j \geq 2$ integers, $a_j \neq 0$, and $gcd(k_j, q) = 1$ for all j. Let $E, F \subset \mathbb{F}_q^d$. Define $\Delta_P(E,F) = \{P(x-y) \in \mathbb{F}_q : x \in E, y \in F\}$. If $|E||F| \gtrsim q^{d+1}$, then $|\Delta_P(E,F)| \gtrsim q$.*

Proof. From Theorem 4.1, it suffices to show that the estimate (4.2) holds. However, this is an immediate result from Weil's theorem (Theorem 2.2) and the proof is complete. $\square$

REMARK 4.3. We stress that Corollary 3.2 does not imply Corollary 4.2 above. Considering the diagonal polynomial $P(x) = \sum_{j=1}^{d} a_j x_j^{k_j}$, if the exponents k_j are distinct, then Corollary 3.2 does not give any information. Authors in this paper have not found any reference which shows that for $m \in \mathbb{F}_q^d \setminus \{(0,\dots,0)\}$, and $t \neq 0$,

$$|\widehat{V_t}(m)| \lesssim q^{-\frac{d+1}{2}},$$

where $V_t = \{x \in \mathbb{F}_q^d : \sum_{j=1}^{d} a_j x_j^{k_j} = t\}$ and all k_j are not same. Thus, we can not apply Theorem 3.1 to obtain such result as in Corollary 4.2. In conclusion, Theorem 4.1 can be very powerful to study the generalized Falconer distance problems. We remark that using some powerful results from algebraic geometry we can find more concrete examples of polynomials satisfying (4.2) or (4.1). For example, see Theorem 8.4 in [4] or Theorem 9.2 in [5].

4.1. Distance problems related to generalized paraboloids. In this subsection, we shall find a useful theorem which yields the simple distance formula in Theorem 4.1. If $E, F \subset \mathbb{F}_q^d$ are product sets with $E = F$ and $P(x) = x_1^2 + \cdots + x_d^2$, then it was proved in [2] that if $|E||F| \gtrsim q^{2d^2/(2d-1)}$, then $|\Delta_P(E,F)| \gtrsim q$. Here, we study the generalized Falconer distance problems for product sets, related to the generalized paraboloid distances which are different from the usual spherical distance. If a distance set is related to usual spheres or paraboloids, then we can take advantage of the explicit forms in the varieties. In these settings, if E and F are product sets in $\mathbb{F}_q^d$, we may easily get the improved Falconer distance result, $|E||F| \gtrsim q^{2d^2/(2d-1)}$. However, if the polynomial generating a distance set is not given in an explicit form, then the generalized distance problem can be hard. We are interested in getting the improved Falconer result on the generalized distance problems for product sets, associated with generalized paraboloids as defined below. Moreover, we aim to apply the result to proving Theorem 4.1. To achieve our aim, we shall work on $\mathbb{F}_q^{d+1}$ instead of $\mathbb{F}_q^d, d \geq 1$. We now introduce the generalized paraboloid in $\mathbb{F}_q^{d+1}$. Given a polynomial $P(x) \in \mathbb{F}_q[x_1,\dots,x_d]$ and $t \in \mathbb{F}_q$, we define the generalized paraboloid $V_t \subset \mathbb{F}_q^{d+1}$ as the set

$$V_t = \{(x, x_{d+1}) \in \mathbb{F}_q^d \times \mathbb{F}_q : P(x) - x_{d+1} = t\}.$$

It is clear that $|V_t| = q^d$ for all $t \in \mathbb{F}_q$, because if we fix $x \in \mathbb{F}_q^d$, then x_{d+1} is uniquely determined. If the polynomial is given by $P(x) = x_1^2 + \cdots + x_d^2$, then V_0 is exactly the usual paraboloid in $\mathbb{F}_q^{d+1}$. Let $H(x, x_{d+1}) = P(x) - x_{d+1}$, where H is a polynomial in $\mathbb{F}_q[x_1,\dots,x_d,x_{d+1}]$. Given $E^*, F^* \subset \mathbb{F}_q^{d+1}$ and $P(x) \in \mathbb{F}_q[x_1,\dots,x_d]$, consider the generalized distance set

$$\Delta_H(E^*, F^*) = \{H(x-y, x_{d+1} - y_{d+1}) \in \mathbb{F}_q : (x, x_{d+1}) \in E^*, (y, y_{d+1}) \in F^*\}.$$

We are interested in the following question. What kinds of conditions on the polynomial $P(x) \in \mathbb{F}_q[x_1,\dots,x_d]$ will allow us to get an improved Falconer exponent for the distance problems associated with the product sets E^* and F^* in $\mathbb{F}_q^{d+1}$? The following theorem provides one condition.

THEOREM 4.4. *Let $P(x) \in \mathbb{F}_q[x_1,\dots,x_d]$ be a polynomial with degree ≥ 2 satisfying the following condition: for each $s \neq 0$ and $m \in \mathbb{F}_q^d$,*

$$\left| \sum_{x \in \mathbb{F}_q^d} \chi(sP(x) + m \cdot x) \right| \lesssim q^{\frac{d}{2}}. \tag{4.3}$$

If $E^* = E \times E_{d+1}$ and $F^* = F \times F_{d+1}$ are product sets in $\mathbb{F}_q^d \times \mathbb{F}_q$, and $\frac{|E^*||F^*|}{|F_{d+1}|} \gtrsim q^{d+1}$, then

$$|\Delta_H(E^*, F^*)| = |\{H(x-y, x_{d+1} - y_{d+1}) \in \mathbb{F}_q : (x, x_{d+1}) \in E^*, (y, y_{d+1}) \in F^*\}| \gtrsim q.$$

Proof. Let $E^*, F^* \subset \mathbb{F}_q^{d+1}$ be product sets given by the forms: $E^* = E \times E_{d+1}$ and $F^* = F \times F_{d+1}$ in $\mathbb{F}_q^d \times \mathbb{F}_q$. In addition, assume that $\frac{|E^*||F^*|}{|F_{d+1}|} \gtrsim q^{d+1}$. Let $x^*, y^* \in \mathbb{F}_q^{d+1}$. As before, consider the counting function ν on $\mathbb{F}_q$ given by

$$\nu(t) = |\{(x^*, y^*) \in E^* \times F^* : H(x^* - y^*) = t\}|.$$

For each $t \in \mathbb{F}_q$, let

$$V_t = \{x^* \in \mathbb{F}_q^{d+1} : H(x^*) - t = 0\}.$$

We are interested in measuring the lower bound of the distance set $\Delta_H(E^*, F^*)$ defined by

$$\Delta_H(E^*, F^*) = \{H(x^* - y^*) \in \mathbb{F}_q : x^* \in E^*, y^* \in F^*\}.$$

In dimension $(d+1)$, applying the Fourier inversion theorem (2.2) to the function $V_t(x^* - y^*)$ and using the definition of the Fourier transforms (2.1), we have

$$\begin{aligned}
\nu(t) =& \sum_{x^* \in E^*, y^* \in F^*} V_t(x^* - y^*) \\
=& q^{2(d+1)} \sum_{m^* \in \mathbb{F}_q^{d+1}} \overline{\widehat{E^*}}(m^*) \widehat{F^*}(m^*) \widehat{V}_t(m^*) \\
=& q^{2(d+1)} \overline{\widehat{E^*}}(0, \ldots, 0) \widehat{F^*}(0, \ldots, 0) \widehat{V}_t(0, \ldots, 0) + q^{2(d+1)} \sum_{m^* \in \mathbb{F}_q^{d+1} \setminus \{(0,\ldots,0)\}} \overline{\widehat{E^*}}(m^*) \widehat{F^*}(m^*) \widehat{V}_t(m^*) \\
=& \frac{|E^*||F^*|}{q} + q^{2(d+1)} \sum_{m^* \in \mathbb{F}_q^{d+1} \setminus \{(0,\ldots,0)\}} \overline{\widehat{E^*}}(m^*) \widehat{F^*}(m^*) \widehat{V}_t(m^*).
\end{aligned}$$

Squaring the $\nu(t)$ and summing it over $t \in \mathbb{F}_q$ yield that

$$\begin{aligned}
\sum_{t \in \mathbb{F}_q} \nu^2(t) =& \frac{|E^*|^2 |F^*|^2}{q} + 2q^{2d+1} |E^*||F^*| \sum_{m^* \in \mathbb{F}_q^{d+1} \setminus \{(0,\ldots,0)\}} \overline{\widehat{E^*}}(m^*) \widehat{F^*}(m^*) \sum_{t \in \mathbb{F}_q} \widehat{V}_t(m^*) \\
&+ q^{4(d+1)} \sum_{m^*, \xi^* \in \mathbb{F}_q^{d+1} \setminus \{(0,\ldots,0)\}} \overline{\widehat{E^*}}(m^*) \widehat{F^*}(m^*) \overline{\widehat{E^*}}(\xi^*) \widehat{F^*}(\xi^*) \sum_{t \in \mathbb{F}_q} \widehat{V}_t(m^*) \widehat{V}_t(\xi^*) \\
=& \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

Observe that I and II are given by

$$\mathrm{I} = \frac{|E^*|^2 |F^*|^2}{q} \quad \text{and } \mathrm{II} = 0, \tag{4.4}$$

where $\mathrm{II} = 0$ follows immediately from the fact that $\sum_{t \in \mathbb{F}_q} \widehat{V}_t(m^*) = 0$ for $m^* \neq (0, \ldots, 0)$. In order to estimate III, first observe that for $m^* = (m, m_{d+1}) \in \mathbb{F}_q^{d+1}$,

$$\widehat{V}_t(m^*) = \frac{1}{q^{d+1}} \sum_{x \in \mathbb{F}_q^d} \chi(-m_{d+1} P(x) - m \cdot x) \chi(t m_{d+1}).$$

It therefore follows that for $m^* = (m, m_{d+1}), \xi^* = (\xi, \xi_{d+1}) \in \mathbb{F}_q^{d+1}$,

$$\widehat{V}_t(m^*) \widehat{V}_t(\xi^*) = \frac{1}{q^{2(d+1)}} \sum_{x, y \in \mathbb{F}_q^d} \chi(t(m_{d+1} + \xi_{d+1})) \chi(-m_{d+1} P(x) - m \cdot x) \chi(-\xi_{d+1} P(y) - \xi \cdot y).$$

Notice that if $m^* \neq (0, \ldots, 0)$ and $m_{d+1} = 0$, then $\widehat{V}_t(m^*)$ vanishes. In addition, observe that if $m_{d+1} + \xi_{d+1} \neq 0$, then $\sum_{t \in \mathbb{F}_q} \widehat{V}_t(m^*) \widehat{V}_t(\xi^*)$ also vanishes and if $m_{d+1} + \xi_{d+1} = 0$, then $\sum_{t \in \mathbb{F}_q} \chi(t(m_{d+1} +$

$\xi_{d+1})) = q$. From these observations together with a change of a variable, $m_{d+1} \to s$, we obtain that

$$\mathrm{III} = q^{2d+3} \sum_{m,\xi \in \mathbb{F}_q^d} \sum_{s \in \mathbb{F}_q \setminus \{0\}} \overline{\widehat{E^*}}(m,s)\widehat{F^*}(m,s)\overline{\widehat{E^*}}(\xi,-s)\widehat{F^*}(\xi,-s)W(m,\xi,s,P),$$

where $W(m,\xi,s,P) = \sum_{x,y\in\mathbb{F}_q^d} \chi(-sP(x) - m\cdot x)\chi(sP(y) - \xi \cdot y)$. Our assumption (4.3) implies that for each $s \neq 0$ and $m, \xi \in \mathbb{F}_q^d$,

$$|W(m,\xi,s,P)| = \left| \sum_{x,y\in\mathbb{F}_q^d} \chi(-sP(x) - m\cdot x)\chi(sP(y) - \xi\cdot y) \right| \lesssim q^d. \tag{4.5}$$

Since $E^* = E \times E_{d+1}$ and $F^* = F \times F_{d+1}$, it is clear that

$$\widehat{E^*}(m,s) = \widehat{E}(m)\widehat{E_{d+1}}(s) \quad \text{and } \widehat{F^*}(m,s) = \widehat{F}(m)\widehat{F_{d+1}}(s).$$

Using this fact along with the inequality (4.5), we see that

$$|\mathrm{III}| \lesssim q^{3(d+1)} \left(\sum_{m\in\mathbb{F}_q^d} \left|\widehat{E}(m)\widehat{F}(m)\right| \right)^2 \left(\sum_{s\in\mathbb{F}_q\setminus\{0\}} \left| \widehat{E_{d+1}}(s)\widehat{F_{d+1}}(s)\right|^2 \right).$$

Using the Cauchy-Schwarz inequality and the trivial bound $|\widehat{F_{d+1}}(s)| \leq |\widehat{F_{d+1}}(0)| = \frac{|F_{d+1}|}{q}$, we obtain that

$$|\mathrm{III}| \lesssim q^{3d+1}|F_{d+1}|^2 \left(\sum_{m\in\mathbb{F}_q^d} \left|\widehat{E}(m)\right|^2 \right) \left(\sum_{m\in\mathbb{F}_q^d} \left|\widehat{F}(m)\right|^2 \right) \left(\sum_{s\in\mathbb{F}_q} \left|\widehat{E_{d+1}}(s)\right|^2 \right).$$

Using the Plancherel theorem (2.3) yields the following:

$$|\mathrm{III}| \lesssim q^d|E||E_{d+1}||F||F_{d+1}|^2 = q^d|E^*||F^*||F_{d+1}|. \tag{4.6}$$

Putting estimates (4.4), (4.6) together, we conclude that

$$\sum_{t\in\mathbb{F}_q} \nu^2(t) \lesssim \frac{|E^*|^2|F^*|^2}{q} + q^d|E^*||F^*||F_{d+1}|.$$

By the Cauchy-Schwarz inequality, we see that

$$\begin{aligned} |E^*|^2|F^*|^2 &= \left(\sum_{t\in\Delta_H(E^*,F^*)} \nu(t) \right)^2 \\ &\leq |\Delta_H(E^*,F^*)| \left(\sum_{t\in\mathbb{F}_q} \nu^2(t) \right). \end{aligned}$$

Thus, we have proved the following:

$$|\Delta_H(E^*,F^*)| \gtrsim \min\left(q, q^{-d}|E^*||F^*||F_{d+1}|^{-1}\right).$$

This implies that if $\frac{|E^*||F^*|}{|F_{d+1}|} \gtrsim q^{d+1}$, then

$$|\Delta_H(E^*,F^*)| \gtrsim q,$$

which completes the proof. $\square$

4.2. Proof of Theorem 4.1. We prove that the general paraboloid distance problem for product sets in $\mathbb{F}_q^{d+1}$ implies the generalized distance problem in $\mathbb{F}_q^d$. Namely, Theorem 4.1 can be obtained as a corollary of Theorem 4.4.

Proof. In order to prove Theorem 4.1, first fix $E, F \subset \mathbb{F}_q^d$ with $|E||F| \geq Cq^{d+1}$ with $C > 0$ large. Let $E^* = E \times \{0\} \subset \mathbb{F}_q^{d+1}$ and $F^* = F \times \{0\} \subset \mathbb{F}_q^{d+1}$. Observe that $|E| = |E^*|, |F| = |F^*|$, and

$$|\Delta_P(E, F)| = |\{P(x-y) \in \mathbb{F}_q : x \in E, y \in F\}|$$

$$= |\Delta_H(E^*, F^*)| = |\{H(x-y, x_{d+1} - y_{d+1}) \in \mathbb{F}_q : (x, x_{d+1}) \in E^*, (y, y_{d+1}) \in F^*\}|$$

where $H(x, x_{d+1}) = P(x) - x_{d+1}$. The assumption (4.2) is the same as (4.3), hence if $\frac{|E^*||F^*|}{|\{0\}|} \gtrsim q^{d+1}$, then $|\Delta_H(E^*, F^*)| \gtrsim q$. Since $|\{0\}| = 1, |E^*| = |E|, |F^*| = |F|$, and $|\Delta_H(E^*, F^*)| = |\Delta_P(E, F)|$, we therefore conclude that if $|E||F| \gtrsim q^{d+1}$, then $|\Delta_P(E, F)| \gtrsim q$. Thus, the proof of Theorem 4.1 is complete. $\square$

5. Generalized pinned distance problems

We find the conditions on the polynomial $P(x) \in \mathbb{F}_q[x_1, \ldots, x_d]$ such that the desirable results for generalized pinned distance problems hold. First, let us introduce some notation associated with the pinned distance problems. Let $P(x) \in \mathbb{F}_q[x_1, \ldots, x_d]$ be a polynomial. For each $t \in \mathbb{F}_q$, we define a variety V_t by

$$V_t = \{x \in \mathbb{F}_q^d : P(x) = t\}.$$

The Schwartz-Zippel lemma (Lemma 2.4) says that $|V_t| \lesssim q^{d-1}$ for all $t \in \mathbb{F}_q$. Let $E \subset \mathbb{F}_q^d$. Given $y \in \mathbb{F}_q^d$, we denote by $\Delta_P(E, y)$ a pinned distance set defined as

$$\Delta_P(E, y) = \{P(x-y) \in \mathbb{F}_q : x \in E\}.$$

We are interested in finding the element $y \in \mathbb{F}_q^d$ and the size of $E \subset \mathbb{F}_q^d$ such that $|\Delta_P(E, y)| \gtrsim q$. We have the following theorem.

Theorem 5.1. *Let $T \subset \mathbb{F}_q$ with $|T| \sim 1$. Suppose that the varieties V_t, generated by a polynomial $P(x) \in \mathbb{F}_q[x_1, \cdots, x_d]$, satisfy the following: for all $m \in \mathbb{F}_q^d \setminus \{(0, \cdots, 0)\}$,*

$$|\widehat{V_t}(m)| \lesssim q^{-\frac{d+1}{2}} \quad \text{if } t \notin T \tag{5.1}$$

and

$$|\widehat{V_t}(m)| \lesssim q^{-\frac{d}{2}} \quad \text{if } t \in T. \tag{5.2}$$

Let $E, F \subset \mathbb{F}_q^d$. If $|E||F| \gtrsim q^{d+1}$, then there exists $F_0 \subset F$ with $|F_0| \sim |F|$ such that

$$|\Delta_P(E, y)| \gtrsim q \quad \text{for all } y \in F_0.$$

Proof. Using that $|\Delta_P(E, y)| \leq q$, it suffices to prove that if $|E||F| \gtrsim q^{d+1}$, then

$$\frac{1}{|F|} \sum_{y \in F} |\Delta_P(E, y)| \gtrsim q. \tag{5.3}$$

For each $t \in \mathbb{F}_q$ and $y \in F$, consider the counting function $\nu_y(t)$ given by

$$\nu_y(t) = |\{x \in E : P(x-y) = t\}| = |\{x \in E : x - y \in V_t\}|.$$

Applying the Fourier inversion transform to the function $V_t(x-y)$ and using the definition of the Fourier transform, we see that

$$\begin{aligned}\nu_y(t) &= \sum_{x\in\mathbb{F}_q^d} E(x)V_t(x-y) = q^d \sum_{m\in\mathbb{F}_q^d} \overline{\widehat{E}}(m)\widehat{V_t}(m)\chi(-m\cdot y)\\ &= q^d\overline{\widehat{E}}(0,\dots,0)\widehat{V_t}(0,\dots,0)\chi(0) + q^d \sum_{m\in\mathbb{F}_q^d\setminus\{(0,\dots,0)\}} \overline{\widehat{E}}(m)\widehat{V_t}(m)\chi(-m\cdot y)\\ &= \frac{|E||V_t|}{q^d} + q^d \sum_{m\in\mathbb{F}_q^d\setminus\{(0,\dots,0)\}} \overline{\widehat{E}}(m)\widehat{V_t}(m)\chi(-m\cdot y).\end{aligned}$$

Squaring the $\nu_y(t)$ and summing it over $y\in F$ and $t\in\mathbb{F}_q$, we see that

$$\begin{aligned}\sum_{y\in F}\sum_{t\in\mathbb{F}_q}\nu_y^2(t) &= \sum_{y\in F}\sum_{t\in\mathbb{F}_q}\frac{|E|^2|V_t|^2}{q^{2d}}\\ &+ \sum_{y\in F}\sum_{t\in\mathbb{F}_q} 2|E||V_t| \sum_{m\in\mathbb{F}_q^d\setminus\{(0,\dots,0)\}} \overline{\widehat{E}}(m)\widehat{V_t}(m)\chi(-m\cdot y)\\ &+ \sum_{y\in F}\sum_{t\in\mathbb{F}_q} q^{2d} \sum_{m,\xi\in\mathbb{F}_q^d\setminus\{(0,\dots,0)\}} \overline{\widehat{E}}(m)\widehat{V_t}(m)\chi(-m\cdot y)\overline{\widehat{E}}(\xi)\widehat{V_t}(\xi)\chi(-\xi\cdot y)\\ &= \mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned}$$

Since $|V_t|\lesssim q^{d-1}$ for all $t\in\mathbb{F}_q$, it is clear that

$$|\mathrm{I}| \lesssim \frac{|E|^2|F|}{q}. \tag{5.4}$$

To estimate $|\mathrm{II}|$, first use the definition of the Fourier transform to get

$$|\mathrm{II}| \le 2q^d|E|\left(\max_{m\in\mathbb{F}_q^d\setminus\{(0,\dots,0)\}}\sum_{t\in\mathbb{F}_q}|V_t||\widehat{V_t}(m)|\right)\sum_{m\in\mathbb{F}_q^d\setminus\{(0,\dots,0)\}}|\overline{\widehat{E}}(m)||\widehat{F}(m)|.$$

From the assumptions, (5.1), (5.2), $|T|\sim 1$, and the fact that $|V_t|\lesssim q^{d-1}$ for all $t\in\mathbb{F}_q$, we see that the maximum value term is $\lesssim q^{(d-1)/2}$. If we use the Cauchy-Schwarz inequality and the Plancherel theorem, then we also see that

$$\sum_{m\in\mathbb{F}_q^d\setminus\{(0,\dots,0)\}}|\overline{\widehat{E}}(m)||\widehat{F}(m)| \le \frac{|E|^{\frac12}|F|^{\frac12}}{q^d}.$$

Therefore, the value II can be estimated by

$$|\mathrm{II}| \lesssim q^{\frac{d-1}{2}}|E|^{\frac32}|F|^{\frac12}. \tag{5.5}$$

Now we estimate the value III. We first observe that

$$\begin{aligned}&\sum_{m,\xi\in\mathbb{F}_q^d\setminus\{(0,\dots,0)\}} \overline{\widehat{E}}(m)\widehat{V_t}(m)\chi(-m\cdot y)\overline{\widehat{E}}(\xi)\widehat{V_t}(\xi)\chi(-\xi\cdot y)\\ &\quad= \left|\sum_{m\in\mathbb{F}_q^d\setminus\{(0,\dots,0)\}} \overline{\widehat{E}}(m)\widehat{V_t}(m)\chi(-m\cdot y)\right|^2\end{aligned}$$

which is always a nonnegative real number. Therefore we can bound the term III by expanding the sum over $y\in F$ to the sum over $y\in\mathbb{F}_q^d$. We then sum over $y\in\mathbb{F}_q^d$ first and use the orthogonality relation of the canonical additive character χ to get $m=-\xi$, otherwise the sum is 0. It therefore follows that

$$|\mathrm{III}| \le q^{3d} \sum_{m\in\mathbb{F}_q^d\setminus\{(0,\ldots,0)\}} \sum_{t\in\mathbb{F}_q} |\widehat{V_t}(m)|^2 |\widehat{E}(m)|^2$$
$$\le q^{3d}\left(\max_{m\in\mathbb{F}_q^d\setminus\{(0,\ldots,0)\}} \sum_{t\in\mathbb{F}_q} |\widehat{V_t}(m)|^2\right) \sum_{m\in\mathbb{F}_q^d} |\widehat{E}(m)|^2.$$

Using the Plancherel theorem and the assumption on the Fourier decay of V_t, we see that

$$\sum_{m\in\mathbb{F}_q^d} |\widehat{E}(m)|^2 = \frac{|E|}{q^d} \quad \text{and} \quad \max_{m\in\mathbb{F}_q^d\setminus\{(0,\ldots,0)\}} \sum_{t\in\mathbb{F}_q} |\widehat{V_t}(m)|^2 \lesssim q^{-d}.$$

Putting these facts together yields the upper bound of the value $|\mathrm{III}|$:

$$|\mathrm{III}| \lesssim q^d |E|. \tag{5.6}$$

From (5.4), (5.5), and (5.6), we obtain the following estimate:

$$\sum_{t\in F}\sum_{t\in\mathbb{F}_q} \nu_y^2(t) \lesssim \frac{|E|^2|F|}{q} + q^{\frac{d-1}{2}} |E|^{\frac{3}{2}} |F|^{\frac{1}{2}} + q^d|E|.$$

Observe that if $|E||F| \ge Cq^{d+1}$ for $C>0$ sufficiently large, then

$$\sum_{y\in F}\sum_{t\in\mathbb{F}_q} \nu_y^2(t) \lesssim \frac{|E|^2|F|}{q}. \tag{5.7}$$

We are ready to finish the proof. For each $y\in F$, we note that $\sum_{t\in\Delta_P(E,y)} \nu_y(t) = |E|$ and apply the Cauchy-Schwarz inequality, then we have

$$|E|^2|F|^2 = \left(\sum_{y\in F}\sum_{t\in\Delta_P(E,y)} \nu_y(t)\right)^2 \le \left(\sum_{y\in F} |\Delta_P(E,y)|\right)\left(\sum_{y\in F}\sum_{t\in\mathbb{F}_q} \nu_y^2(t)\right)$$
$$\lesssim \left(\sum_{y\in F} |\Delta_P(E,y)|\right) \frac{|E|^2|F|}{q}$$

where the last line follows from the estimate (5.7). Thus, the estimate (5.3) holds and we complete the proof of Theorem 5.1. ☐

Remark 5.2. Let $E, F\subset \mathbb{F}_q^d$ satisfy the assumptions of Corollary 3.2. We note that $P(x_1,..,x_d) = a_1x_1^k+\cdots+a_dx_d^k$ satisfies the assumptions of Corollary 3.2, therefore there exists a subset F_0 of F with $|F_0|\sim|F|$ such that

$$|\Delta_P(E,y)| \gtrsim q \quad \text{for all } y\in F_0.$$

This is an immediate result from Theorem 5.1 and Lemma 2.1. In terms of the generalized Falconer distance problem, this result sharpens the statement of Corollary 3.2. On the other hand, Corollary 3.2 gives us the exact number of the elements in the distance set.

We close this paper by introducing a corollary of Theorem 5.1, which sharpens and generalizes Vu's result (1.3).

Corollary 5.3. *Let $P(x)\in\mathbb{F}_q[x_1,x_2]$ be a nondegenerate polynomial. If $|E||F|\gtrsim q^3$ for $E,F\subset\mathbb{F}_q^2$, then there exists a subset F_0 of F with $|F_0|\sim|F|$ such that*

$$|\Delta_P(E,y)| \gtrsim q \quad \textit{for all } y\in F_0.$$

Proof. The proof follows immediately by applying Theorem 5.1 along with Theorem 2.5 and (2.4) in Remark 2.6. $\square$

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