Restriction operators acting on radial functions on vector spaces over finite fields

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Abstract. We stduy $L^p - L^r$ restriction estimates for algebraic varieties V in the case when restriction operators act on radial functions in the finite field setting. We show that if the varieties V lie in odd dimensional vector spaces over finite fields, then the conjectured restriction estimates are possible for all radial test functions. In addition, it is proved that if the varieties V in even dimensions have few intersection points with the sphere of zero radius, the same conclusion as in odd dimensional case can be also obtained.

1. Introduction

Let V be a subset of $\mathbb{R}^d, d \geq 2$, and $d\sigma$ a positive measure supported on V. Then, one may ask that for which values of p and r does the following inequality

$$\|\widehat{f}\|_{L^r(V,d\sigma)} \leq C_{p,r,d} \, \|f\|_{L^p(\mathbb{R}^d)} \quad \text{for all} \quad f \in L^p(\mathbb{R}^d)$$

hold? This problem is known as the restriction problem in Euclidean space and it was first posed by E.M. Stein in 1967. The restriction problem for the circle and the parabola in the plane was completely solved by Zygmund ([**16**]) and the problem for cones in three and four dimesions was also established by Barcelo ([**1**]) and Wolff ([**14**]) respectively. However, this problem is still open in other higher dimensions and it have been considered as one of the most important, difficult problems in harmonic analysis. We refer the reader to [**13**],[**15**], [**11**],[**2**], [**3**], [**12**] for further discussion and recent progress on the restriction problem.

As an analog of the Euclidean restriction problem, Tao and Mockenhaupt ([**10**]) recently reformulated and studied the restriction problem for various algebraic varieties in the finite field setting. In this introduction we review the definition, a conjecture, and known results on the restriction problem for algebraic varieties in d-dimensional vector spaces over finite fields. Let $\mathbb{F}_q^d$ be a d-dimensional vector space over the finite field $\mathbb{F}_q$ with q elements. We endow this space with a counting measure dm. Thus, if $f : \mathbb{F}_q^d \to \mathbb{C}$, then its integral over $\mathbb{F}_q^d$ is given by

$$\int_{\mathbb{F}_q^d} f(m)dm = \sum_{m\in\mathbb{F}_q^d} f(m).$$

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We denote by $\mathbb{F}_{q*}^d$ the dual space of $\mathbb{F}_q^d$. We endow the dual space $\mathbb{F}_{q*}^d$ with a normalized counting measure dx. Hence, given a function $g : \mathbb{F}_{q*}^d \to \mathbb{C}$, we define its integral

$$\int_{\mathbb{F}_{q*}^d} g(x)dx = \frac{1}{q^d} \sum_{x \in \mathbb{F}_{q*}^d} g(x).$$

Recall that the space $\mathbb{F}_q^d$ is isomorphic to its dual space $\mathbb{F}_{q*}^d$ as an abstract group. Also recall that if f is a complex-valued function on $\mathbb{F}_q^d$, its Fourier transform, denoted by $\widehat{f}$, is actually defined on its dual space $\mathbb{F}_{q^*}^d$:

$$\widehat{f}(x) = \int_{\mathbb{F}_q^d} \chi(-m \cdot x) f(m) dm = \sum_{m \in \mathbb{F}_q^d} \chi(-m \cdot x) f(m),$$

where χ denotes a nontrivial additive character of $\mathbb{F}_q$. Let V be an algebraic variety in the dual space $\mathbb{F}_{q*}^d$. Throughtout the paper we always assume that $|V| \sim q^{d-1}$. Namely, we view the variety V as a hypersurface in $\mathbb{F}_{q*}^d$. Recall that a normalized surface measure on V, denoted by $d\sigma$, is defined by the relation

$$\int g(x) d\sigma(x) = \frac{1}{|V|} \sum_{x \in V} g(x),$$

where $g : \mathbb{F}_{q*}^d \to \mathbb{C}$. With notation above, the restriction problem for the variety V is to determine $1 \leq p, r \leq \infty$ such that the following restriction estimate holds:

$$\|\widehat{f}\|_{L^r(V,d\sigma)} \leq C \|f\|_{L^p(\mathbb{F}_q^d, dm)} \quad \text{for all functions } f : \mathbb{F}_q^d \to \mathbb{C}, \tag{1.1}$$

where the constant $C > 0$ is independent of functions f and the size of the underlying finite field $\mathbb{F}_q$. We shall use the notation $R(p \to r) \lesssim 1$ to indicate that the restriction estimate (1.1) holds. By duality, the inequality (1.1) is same as the following extension estimate:

$$\|(g d\sigma)^\vee\|_{L^{p'}(\mathbb{F}_q^d, dm)} \leq C \|g\|_{L^{r'}(V, d\sigma)}.$$

Mockenhaupt and Tao ([**10**]) observed that the necessary conditions for the inequality (1.1) take

$$1 \leq p, r \leq \infty, \quad \frac{1}{p} \geq \frac{d+1}{2d} \quad \text{and} \quad \frac{d}{p} + \frac{d-1}{r} \geq d. \tag{1.2}$$

Namely, $R(p \to r) \lesssim 1$ only if $(1/p, 1/r)$ lies in the convex hull of points

$$(1,0), (1,1), \left(\frac{d+1}{2d}, 1\right), \text{ and } \left(\frac{d+1}{2d}, \frac{1}{2}\right). \tag{1.3}$$

They also proved that the necessary conditions (1.3) are in fact sufficient for $A(p \to r) \lesssim 1$ if V is the parabola in $\mathbb{F}_{q*}^2$. In [**7**], Koh and Shen generalized the result to general algebraic cuves in two dimensions. However, in higher dimensions the restriction problem has not been solved and the known results are even weaker than those in Euclidean space. The currently best known results on restriction problems for paraboloids in $\mathbb{F}_{q*}^d$ are due to A. Lewko and M.Lewko ([**8**]). They established certain endpoint restriction estimates for paraboloids, which slightly improve on the previously known results by Mockenhaupt and Tao([**10**]) in three dimensions and those by Iosevich and Koh ([**4**]) in higher dimensions. More precisely, the following theorem was proved by them.

Theorem 1.1. *Let $V = \{x \in \mathbb{F}_{q^*}^d : x_1^2 + \cdots + x_{d-1}^2 = x_d\}$ be the paraboloid. If $d = 3$ and $-1 \in \mathbb{F}_{q*}$ is not a square, then $A(p \to r) \lesssim 1$ whenever $(1/p, 1/r)$ lies in the convex hull of points*

$$(1,0), (1,1), (13/18, 1), (13/18, 1/2), \text{ and } (3/4, 3/8).$$

Moreover, if $d \geq 4$ is even or if $d = 4k+3$ for some $k \in \mathbb{N}$ and $-1 \in \mathbb{F}_{q*}$ is not a square, then $A(p \to r) \lesssim 1$ whenever $(1/p, 1/r)$ is contained in the convex hull of points

$$(1,0), (1,1), \left(\frac{d^2+2d-2}{2d^2}, 1\right), \left(\frac{d^2+2d-2}{2d^2}, \frac{1}{2}\right), \text{ and } \left(\frac{3}{4}, \frac{d+2}{4d}\right). \tag{1.4}$$

As we shall see, our main results below imply that the restriction conjecture (1.2) for paraboloids in $\mathbb{F}_{q*}^d$ holds if the restriction operator acts on radial functions. The main purpose of this paper provides general properties of varieties for which the restriction conjecture holds for all radial test functions.

2. Statement of main results

While we do not know how to improve Theorem 1.1, we are able to show that if the test functions are radial functions, then the $L^p - L^r$ restriction estimates for paraboloids hold for the conjectured exponents given in (1.3). In fact, we shall prove more strong results. In order to clearly state our main theorems, let us introduce certain definitions and notation. For each $m = (m_1, \ldots, m_d) \in \mathbb{F}_q^d$, define

$$\|m\| = m_1^2 + \cdots + m_d^2.$$

We say that a function $f : \mathbb{F}_q^d \to \mathbb{C}$ is a radial function if

$$f(m) = f(n) \quad \text{whenever } \|m\| = \|n\|.$$

For each $j \in \mathbb{F}_q$, we define

$$S_j^{d-1} = \{m = (m_1, \ldots, m_d) \in \mathbb{F}_q^d : m_1^2 + \cdots + m_d^2 = j\} \tag{2.1}$$

which will be named as the sphere with j radius.

Definition 2.1. We write $R_{rad}(p \to r) \lesssim 1$ if the restriction estimate (1.1) holds for all radial functions $f : \mathbb{F}_q^d \to \mathbb{C}$.

2.1. Restriction results on radial functions. Our first result below shows that the restriction operators acting on radial functions have quite good mapping properties.

Theorem 2.2. *Let $d\sigma$ be the normalized surface measure on an algebraic variety $V \subset \mathbb{F}_{q*}^d$ with $|V| \sim q^{d-1}$. Then, we have*

$$R_{rad}\left(\frac{2d}{d+1} \to 2\right) \lesssim 1 \quad \textit{for } d \geq 3 \textit{ odd} \tag{2.2}$$

and

$$R_{rad}\left(\frac{2d-2}{d} \to \frac{2(d-1)^2}{d^2-2d}\right) \lesssim 1 \quad \textit{for } d \geq 4 \textit{ even.} \tag{2.3}$$

Using the nesting properties of L^p-norms and interpolating (2.2) with the trivial $L^1 - L^\infty$, we see that the necessary conditions (1.2) for $A(p \to r) \lesssim 1$ are sufficient for $R_{rad}(p \to r) \lesssim 1$ if the variety V with $|V| \sim q^{d-1}$ lies in odd dimensional vector spaces finite fields. Notice that the result of Theorem 2.1 in even dimensions is weaker than that in odd dimensions. However, the following theorem shows that if the variety V does not contain a lot of elements in the sphere with zero radius, then the result in even dimensions can be improved to that in odd dimensions.

THEOREM 2.3. *Let $d\sigma$ be the normalized surface measure on an algebraic variety $V \subset \mathbb{F}_{q*}^d$ with $|V| \sim q^{d-1}$. Suppose that $|V \cap S_0^{d-1}| \lesssim q^{\frac{d^2-d-1}{d}}$. Then*

$$R_{rad}\left(\frac{2d}{d+1} \to 2\right) \lesssim 1 \quad \text{for } d \geq 3.$$

It seems that if the algebraic vareity V does not contain S_0^{d-1}, the sphere of zero radius, then the conclusion of Theorem 2.3 holds. For example, if $V = \{x \in \mathbb{F}_{q*}^d : x_1^2 + \cdots + x_{d-1}^2 = x_d\}$ is the paraboloid or $V = \{x \in \mathbb{F}_{q*}^d : x_1 + \cdots + x_d = 0\}$ is the plane, then $|V \cap S_0^{d-1}| \lesssim q^{d-2} < q^{\frac{d^2-d-1}{d}}$. In this case, we threfore obtain the conclusion of Theorem 2.3. This fact is very interesting in that the Fourier transform of radial functions can be meaningfully restricted to the plane.

3. Fourier decay estimates on spheres

Since the Fourier transform of a radial function can be written as a linear combination of the Fourier transforms on spheres, the Fourier decay estimates on spheres shall play a crucial role in proving our results. In this section, we go over the decay properties of the Fourier transform on spheres S_j in $\mathbb{F}_q^d$. We begin with reviewing the classical exponential sum estimates. For each $a \in \mathbb{F}_q^*$, the Gauss sum is defined by

$$G_a := \sum_{s \in \mathbb{F}_q^*} \eta(s)\chi(as),$$

where η denotes the quadratic character of $\mathbb{F}_q^*$. The Kloosterman sum is given by

$$K(a,b) := \sum_{s \in \mathbb{F}_q^*} \chi(as + bs^{-1}) \quad \text{for } a, b \in \mathbb{F}_q.$$

In addition, recall that the Salié sum is the exponential sum given by

$$S(a,b) := \sum_{s \in \mathbb{F}_q^*} \eta(s)\chi(as + bs^{-1}) \quad \text{for } a, b \in \mathbb{F}_q.$$

It is well known that $|G_a| = \sqrt{q}$ for $a \in \mathbb{F}_q^*$, $|K(a,b)| \leq 2\sqrt{q}$ for $ab \neq 0$, and $|S(a,b)| \leq 2\sqrt{q}$ for $a, b \in \mathbb{F}_q$ (see p.193 in [**9**] and pp.322-323 in [**6**]). In terms of the aforementioned exponential sums, the authors in [**5**] expressed the Fourier transforms on the spheres in $\mathbb{F}_{q^*}^d$ with the normalized counting measure dx. Modifying the normalizing factor, one can also do the same thing for the Fourier transform on the spheres in the space $\mathbb{F}_q^d$ with the counting measure dm.

Lemma 3.1. *Let S_j^{d-1} be the sphere in $\mathbb{F}_q^d$ defined as in (2.1). Then for any $x \in \mathbb{F}_{q*}^d$, we have*

$$\widehat{S_j^{d-1}}(x) = \begin{cases} q^{d-1}\delta_0(x) + q^{-1}G^d K(-j, -4^{-1}\|x\|) & \text{for } d \geq 2 \quad \text{even} \\ q^{d-1}\delta_0(x) + q^{-1}G^d S(-j, -4^{-1}\|x\|) & \text{for } d \geq 3 \quad \text{odd} \end{cases}$$

where $\delta_0(x) = 1$ if $x = (0, \ldots, 0)$ and $\delta_0(x) = 0$ otherwise.

Proof. We follows the same arguments as in Lemma 4 in [**5**]. By the definition of the Fourier transform of a function on $\mathbb{F}_q^d$ with the counting measure dm, if $S_j^{d-1} \subset \mathbb{F}_q$ and $x \in \mathbb{F}_{q^*}^d$, then

$$\widehat{S_j^{d-1}}(x) = \sum_{m \in S_j^{d-1}} \chi(-x \cdot m) = \sum_{m \in \mathbb{F}_q^d} \chi(-x \cdot m)\delta_0(\|m\| - j).$$

Applying the orthogonality relation of χ, we can write

$$\delta_0(\|m\| - j) = q^{-1} \sum_{s \in \mathbb{F}_q} \chi(s(\|m\| - j)) \quad \text{for } x \in \mathbb{F}_{q^*}^d.$$

It therefore follows that

$$\widehat{S_j^{d-1}}(x) = q^{-1} \sum_{m \in \mathbb{F}_q^d} \chi(-m \cdot x) + q^{-1} \sum_{s \in \mathbb{F}_q^*} \chi(-js) \left(\sum_{m \in \mathbb{F}_q^d} \chi(s\|m\| - x \cdot m) \right)$$

$$(3.1) \qquad = q^{d-1}\delta_0(x) + q^{-1} \sum_{s \in \mathbb{F}_q^*} \chi(-js) \prod_{k=1}^{d} \sum_{m_k \in \mathbb{F}_q} \chi(sm_k^2 - x_k m_k).$$

Completing the square and using a change of variables, it follows that

$$(3.2) \qquad \sum_{m_k \in \mathbb{F}_q} \chi(sm_k^2 - x_k m_k) = \chi(-x_k^2/(4s^2)) \sum_{m_k \in \mathbb{F}_q} \chi(sm_k^2) \quad \text{for } k = 1, 2, \ldots, d.$$

Let $A = \{t \in \mathbb{F}_q^* : t \text{ is a square number}\}$ and observe that for each $s \in \mathbb{F}_q^*$,

$$\sum_{t \in \mathbb{F}_q} \chi(st^2) = 1 + \sum_{t \in \mathbb{F}_q^*} \chi(st^2) = 1 + \sum_{t \in A} 2\chi(st)$$

$$= 1 + \sum_{t \in \mathbb{F}_q^*} (1 + \eta(t))\chi(st) = \sum_{t \in \mathbb{F}_q^*} \eta(t)\chi(st) = \eta(s)G_1.$$

Applying this equality to (3.2), it follows from the equality (3.1) that

$$\widehat{S_j^{d-1}}(x) = q^{d-1}\delta_0(x) + q^{-1}G^d \sum_{s \in \mathbb{F}_q^*} \eta^d(s)\chi\left(-js + \frac{\|x\|}{-4s}\right).$$

Since $\eta^d = 1$ for $d \geq 2$ even, and $\eta^d = \eta$ for $d \geq 3$ odd, the statement of Lemma 3.1 follows immediately from the definitions of the Kloosterman sum and the Salié sum. $\square$

The following corollary can be obtained by applying the estimates of the Gauss sum G, the Kloosterman $K(a, b)$, and the Salié sum $S(a, b)$ to Lemma 3.1.

Corollary 3.2. *Let $d \geq 3$ be an integer. Then,*

$$\widehat{S_j^{d-1}}(0, \ldots, 0) = |S_j^{d-1}| \sim q^{d-1} \quad \text{for } j \in \mathbb{F}_q. \tag{3.3}$$

If $d \geq 2$ and $x \in \mathbb{F}_{q*}^d \setminus \{(0, \ldots, 0)\}$ then

$$|\widehat{S_j^{d-1}}(x)| \lesssim \begin{cases} q^{\frac{d-1}{2}} & \text{for } d \text{ odd}, \ j \in \mathbb{F}_q \\ q^{\frac{d-1}{2}} & \text{for } d \text{ even}, \ j \neq 0 \\ q^{\frac{d}{2}} & \text{for } d \text{ even}, \ j = 0. \end{cases} \tag{3.4}$$

In particular, if $\|x\| \neq 0$ and $d \geq 4$ is even, then

$$|\widehat{S_0^{d-1}}(x)| = q^{\frac{d-2}{2}}. \tag{3.5}$$

Proof. First, let us prove (3.3). It is clear from the definition of the Fourier transform that $\widehat{S_j^{d-1}}(0, \ldots, 0) = \sum_{m \in \mathbb{F}_q^d} \chi(m \cdot (0, \ldots, 0)) S_j^{d-1}(m) = |S_j^{d-1}|$. On the other hand, it follows from Lemma 3.1 that

$$\widehat{S_j^{d-1}}(0, \ldots, 0) = \begin{cases} q^{d-1} + q^{-1} G^d K(-j, 0) & \text{for } d \geq 2 \quad \text{even} \\ q^{d-1} + q^{-1} G^d S(-j, 0) & \text{for } d \geq 3 \quad \text{odd} \end{cases}$$

Since $|G| = \sqrt{q}$, $|K(-j, 0)| \leq q$ for $j \in \mathbb{F}_q$, and $|S(-j, 0)| \leq 2\sqrt{q}$ for $j \in \mathbb{F}_q$, we see that if $d \geq 3$, then $q^{d-1} + q^{-1} G^d K(-j, 0) \sim q^{d-1} + q^{-1} G^d S(-j, 0) \sim q^{d-1}$. Thus (3.3) holds. Next, using Lemma 3.1, the conclusion (3.4) is an immdeate consequence from facts that $|G| \sqrt{q}$, $|K(a, b)| \leq 2\sqrt{q}$ if $ab \neq 0$, $|K(a, b)| \leq q$ if $ab = 0$, and $|S(a, b)| \leq 2\sqrt{q}$ if $a, b \in \mathbb{F}_q$. Finally, the equality (3.5) follows from Lemma 3.1 and the observations that $|G| = \sqrt{q}$, $|K(0, b)| = 1$ for $b \neq 0$. $\square$

4. Proofs of Theorem 2.1 and Theorem 2.3

In this section we shall complete the proofs of main results on restriction operators acting on radial functions. First we will derive sufficient conditions for $R_{rad}(p \to r) \lesssim 1$. We aim to find certain conditions on $1 \leq p, r \leq \infty$ such that

$$\|\widehat{f}\|_{L^r(V, d\sigma)} \lesssim \|f\|_{L^p(\mathbb{F}_q^d, dm)} \quad \text{for all radial functions } f : \mathbb{F}_q^d \to \mathbb{C}.$$

Without loss of generality, we may assume that f is a nonnegative, radial function on $\mathbb{F}_q^d$. Therefore, we can write

$$f(m) = M_j \geq 0 \quad \text{if } m \in S_j^{d-1} \text{ for some } j \in \mathbb{F}_q.$$

By multiplying a normalizing constant, we may also assume that

$$\|f\|_{L^p(\mathbb{F}_q^d, dm)} = 1.$$

It therefore follows that

$$1 = \|f\|_{L^p(\mathbb{F}_q^d, dm)}^p = \sum_{m \in \mathbb{F}_q^d} |f(m)|^p = \sum_{j \in \mathbb{F}_q} \sum_{m \in S_j^{d-1}} M_j^p = \sum_{j \in \mathbb{F}_q} M_j^p |S_j^{d-1}|$$

Since $|S_j^{d-1}| \sim q^{d-1}$ for $j \in \mathbb{F}_q$, we have

$$\sum_{j \in \mathbb{F}_q} M_j^p \sim q^{1-d}. \tag{4.1}$$

With assumptions above on the radial function f, it suffices to find certain conditions on $1\le p,r\le\infty$ such that

$$\|\widehat{f}\|^r_{L^r(V,d\sigma)} := \frac{1}{|V|}\sum_{x\in V}|\widehat{f}(x)|^r \lesssim 1. \tag{4.2}$$

Since $\widehat{f}(x)=\sum\limits_{m\in\mathbb{F}_q^d}\chi(-m\cdot x)f(m)=\sum\limits_{j\in\mathbb{F}_q}\sum\limits_{m\in S_j^{d-1}}\chi(-m\cdot x)M_j$, It follows that

$$\|\widehat{f}\|^r_{L^r(V,d\sigma)} = \frac{1}{|V|}\sum_{x\in V}\left|\sum_{j\in\mathbb{F}_q}M_j\widehat{S}_j(x)\right|^r$$

$$=\frac{1}{|V|}\sum_{x\in V\setminus\{(0,\ldots,0)\}}\left|\sum_{j\in\mathbb{F}_q}M_j\widehat{S_j^{d-1}}(x)\right|^r+\frac{1}{|V|}\left|\sum_{j\in\mathbb{F}_q}M_j\widehat{S_j^{d-1}}(0,\ldots,0)\right|^r.$$

Since $M_j\ge 0, |V|\sim q^{d-1}$, and $\widehat{S_j^{d-1}}(0,\ldots,0)=|S_j^{d-1}|\sim q^{d-1}$ for $d\ge 3$, we see that

$$\|\widehat{f}\|^r_{L^r(V,d\sigma)} \sim \frac{1}{q^{d-1}}\sum_{x\in V\setminus\{(0,\ldots,0)\}}\left|\sum_{j\in\mathbb{F}_q}M_j\widehat{S_j^{d-1}}(x)\right|^r+\frac{q^{r(d-1)}}{q^{d-1}}\left(\sum_{j\in\mathbb{F}_q}M_j\right)^r.$$

From Hölder's inequality and (4.1), observe that

$$\left(\sum_{j\in\mathbb{F}_q}M_j\right)^r \le \left(\sum_{j\in\mathbb{F}_q}1^{p'}\right)^{\frac{r}{p'}}\left(\sum_{j\in\mathbb{F}_q}M_j^p\right)^{\frac{r}{p}} \sim q^{r(1-\frac{d}{p})}, \tag{4.3}$$

where p' denotes the Hölder conjugate of p , that is $p'=p/(p-1)$. It follows that

$$\|\widehat{f}\|^r_{L^r(V,d\sigma)} \lesssim \frac{1}{q^{d-1}}\sum_{x\in V\setminus\{(0,\ldots,0)\}}\left|\sum_{j\in\mathbb{F}_q}M_j\widehat{S_j^{d-1}}(x)\right|^r+q^{rd(1-\frac{1}{p})-d+1}.$$

Combining this with (4.2), the sufficiet conditions on $1\le p,r\le\infty$ for $R_{rad}(p\to r)\lesssim 1$ are given by

$$\frac{1}{q^{d-1}}\sum_{x\in V\setminus\{(0,\ldots,0)\}}\left|\sum_{j\in\mathbb{F}_q}M_j\widehat{S_j^{d-1}}(x)\right|^r\lesssim 1 \tag{4.4}$$

and

$$rd(1-\frac{1}{p})-d+1\le 0. \tag{4.5}$$

4.1. Proof of the first part of conclusions in Theorem 2.1. We prove (2.2) of Theorem 2.1. Namely, we shall prove that if $d\ge 3$ is odd, then

$$\|\widehat{f}\|_{L^2(V,d\sigma)}\lesssim \|f\|_{L^{\frac{2d}{d+1}}(\mathbb{F}_q^d,dm)}\quad \text{for all radial functions } f:\mathbb{F}_q^d\to\mathbb{C}.$$

With $p=\frac{2d}{d+1}$ and $r=2$, it is enough to show that the inequalities (4.4), (4.5) hold. The inequaltiy (4.5) follows immediately from a simple calculation that if $p=\frac{2d}{d+1}$ and $r=2$, then $rd(1-\frac{1}{p})-d+1=0$. To prove the inequality (4.4), recall from (3.4) in Corollary 3.2 that if $d\geq 3$ is odd, then

$$|\widehat{S_j^{d-1}}(x)| \lesssim q^{\frac{d-1}{2}} \quad \text{for } j\in \mathbb{F}_q,\ x\neq (0,\dots,0).$$

From this fact and (4.3), the inequality (4.4) can be established for $p=\frac{2d}{d+1}$ and $r=2$ by the following observation:

$$\frac{1}{q^{d-1}}\sum_{x\in V\setminus\{(0,\dots,0)\}}\left|\sum_{j\in\mathbb{F}_q} M_j\widehat{S_j^{d-1}}(x)\right|^2 \lesssim \sum_{x\in V\setminus\{(0,\dots,0)\}}\left(\sum_{j\in\mathbb{F}_q} M_j\right)^2 \tag{4.6}$$

$$\lesssim |V|\left(\sum_{j\in\mathbb{F}_q} M_j\right)^2 \lesssim q^{d-1}q^{2(1-\frac{d+1}{2})}=1,$$

where we also used that $M_j\geq 0$ and $|V|\sim q^{d-1}$.

4.2. Proof of the second part of conclusions in Theorem 2.1. We prove (2.3) of Theorem 2.1. When $d\geq 4$ is even, we must prove $R_{rad}(p\to r)\lesssim 1$ for $p=\frac{2d-2}{d}$ and $r=\frac{2(d-1)^2}{d^2-2d}$. As mentioned before, it suffices to prove the inequalities (4.4), (4.5) for $p=\frac{2d-2}{d}$ and $r=\frac{2(d-1)^2}{d^2-2d}$. In this case, the inequality (4.5) is clearly true because $rd(1-\frac{1}{p})-d+1=0$. To prove the inequality (4.4), recall from (3.4) in Corollary 3.2 that if $d\geq 4$ is even and $x\neq(0,\dots,0)$, then

$$|\widehat{S_j^{d-1}}(x)| \lesssim \begin{cases} q^{\frac{d-1}{2}} & \text{for } j\neq 0\\ q^{\frac{d}{2}} & \text{for } j=0.\end{cases}$$

From this fact, the left part of the inequality (4.4) can be estimated as follows.

$$\frac{1}{q^{d-1}}\sum_{x\in V\setminus\{(0,\dots,0)\}}\left|\sum_{j\in\mathbb{F}_q} M_j\widehat{S_j^{d-1}}(x)\right|^r$$

$$\lesssim \frac{1}{q^{d-1}}\sum_{x\in V\setminus\{(0,\dots,0)\}}\left|M_0\widehat{S_0^{d-1}}(x)\right|^r + \frac{1}{q^{d-1}}\sum_{x\in V\setminus\{(0,\dots,0)\}}\left|\sum_{j\neq 0} M_j\widehat{S_j^{d-1}}(x)\right|^r$$

$$\lesssim \frac{1}{q^{d-1}}q^{\frac{rd}{2}}M_0^r\left(\sum_{x\in V\setminus\{(0,\dots,0)\}}1\right) + \frac{1}{q^{d-1}}q^{\frac{r(d-1)}{2}}\left(\sum_{j\neq 0}M_j\right)^r\left(\sum_{x\in V\setminus\{(0,\dots,0)\}}1\right)$$

$$\leq q^{\frac{rd}{2}}M_0^r + q^{\frac{r(d-1)}{2}}\left(\sum_{j\in\mathbb{F}_q}M_j\right)^r \lesssim q^{\frac{rd}{2}}M_0^r + q^{\frac{r(d-1)}{2}}q^{r(1-\frac{d}{p})}$$

where the last inequality follows from (4.3). Since $M_j\geq 0$ for $j\in\mathbb{F}_q$, it is clear from (4.1) that $M_0\lesssim q^{\frac{1-d}{p}}$. Thus, we have

$$M_0^r \lesssim q^{\frac{r(1-d)}{p}}. \tag{4.7}$$

We threrefore see that if $p=\frac{2d-2}{d}$ and $r=\frac{2(d-1)^2}{d^2-2d}$, then

$$\frac{1}{q^{d-1}}\sum_{x\in V\setminus\{(0,\ldots,0)\}}\left|\sum_{j\in\mathbb{F}_q} M_j\widehat{S_j^{d-1}}(x)\right|^r \lesssim q^{\frac{rd}{2}+\frac{r(1-d)}{p}}+q^{\frac{r(d-1)}{2}+r(1-\frac{d}{p})}=q^0+q^{-\frac{d-1}{d(d-2)}}\lesssim 1,$$

which proves the inequality (4.4). Hence, we complete the proof.

4.3. Proof of Theorem 2.3. Let $d\sigma$ be the normalized surface measure on an algebraic variety $V\subset\mathbb{F}_{q*}^d$ with $|V|\sim q^{d-1}$. Assuming that $|V\cap S_0^{d-1}|\lesssim q^{\frac{d^2-d-1}{d}}$, we aim to prove that

$$R_{rad}\left(\frac{2d}{d+1}\to 2\right)\lesssim 1 \quad \text{for } d\geq 3.$$

In the case when $d\geq 3$ is odd, this statement was already proved in the first part of Theorem 2.1 with much weaker assumptions. Thus, we may assume that $d\geq 4$ is even. Suppose that

$$|V\cap S_0^{d-1}|\lesssim q^{\frac{d^2-d-1}{d}}. \tag{4.8}$$

As before, our task is to prove that the inequalities (4.4), (4.5) hold for $p=\frac{2d}{d+1}$ and $r=2$. As mentioned before, the inequality (4.5) clearly holds for $p=\frac{2d}{d+1}$ and $r=2$. To prove the inequality (4.4), let

$$\mathrm{L}:=\frac{1}{q^{d-1}}\sum_{x\in V\setminus\{(0,\ldots,0)\}}\left|\sum_{j\in\mathbb{F}_q} M_j\widehat{S_j^{d-1}}(x)\right|^r$$

and show that $L\lesssim 1$. It follows that

$$\mathrm{L}\lesssim\frac{1}{q^{d-1}}\sum_{x\in V\setminus\{(0,\ldots,0)\}} M_0^r\left|\widehat{S_0^{d-1}}(x)\right|^r+\frac{1}{q^{d-1}}\sum_{x\in V\setminus\{(0,\ldots,0)\}}\left|\sum_{j\neq 0} M_j\widehat{S_j^{d-1}}(x)\right|^r:=\mathrm{R}+\mathrm{M}.$$

It suffices to prove that for $p=\frac{2d}{d+1}$ and $r=2$,

$$\mathrm{R}=\frac{1}{q^{d-1}}\sum_{x\in V\setminus\{(0,\ldots,0)\}} M_0^r\left|\widehat{S_0^{d-1}}(x)\right|^r\lesssim 1 \tag{4.9}$$

and

$$\mathrm{M}=\frac{1}{q^{d-1}}\sum_{x\in V\setminus\{(0,\ldots,0)\}}\left|\sum_{j\neq 0} M_j\widehat{S_j^{d-1}}(x)\right|^r\lesssim 1. \tag{4.10}$$

The inequality (4.10) follows immediately from the same argument in (4.6). To prove the inequality (4.9), we write

$$\mathrm{R}=\frac{1}{q^{d-1}}\sum_{x\in V\setminus\{(0,\ldots,0)\}:\|x\|=0} M_0^r\left|\widehat{S_0^{d-1}}(x)\right|^r+\frac{1}{q^{d-1}}\sum_{x\in V\setminus\{(0,\ldots,0)\}:\|x\|\neq 0} M_0^r\left|\widehat{S_0^{d-1}}(x)\right|^r.$$

Since $d\geq 4$ is even, the application of (3.4) and (3.5) in Corollary 3.2 yield that

$$\mathrm{R}\lesssim\frac{M_0^r}{q^{d-1}}q^{\frac{rd}{2}}|V\cap S_0^{d-1}|+\frac{M_0^r}{q^{d-1}}q^{\frac{r(d-2)}{2}}|V|.$$

By (4.7) and our assumption that $|V \cap S_0^{d-1}| \lesssim q^{\frac{d^2-d-1}{d}}$, we see that if $p = \frac{2d}{d+1}$ and $r = 2$, then

$$\mathrm{R} \lesssim q^{\frac{r(1-d)}{p} + \frac{rd}{2} - \frac{1}{d}} + q^{\frac{r(1-d)}{p} + \frac{r(d-2)}{2}} = q^0 + q^{\frac{1-2d}{d}} \lesssim 1.$$

The proof of Theorem 2.3 is complete.

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