

# THE ERDÖS-FALCONER DISTANCE PROBLEM, EXPONENTIAL SUMS, AND FOURIER ANALYTIC APPROACH TO INCIDENCE THEOREMS IN VECTOR SPACES OVER FINITE FIELDS\*

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**Abstract.** We study the Erdős/Falconer distance problem in vector spaces over finite fields with respect to the cubic metric. Estimates for discrete Airy sums and Adolphson/Sperber estimates for exponential sums in terms of Newton polyhedra play a crucial role. Similar techniques are used to study the incidence problem between points and cubic and quadratic curves. As a result we obtain a non-trivial range of exponents that appear to be difficult to attain using combinatorial methods.

**Key words.** distance sets, Newton diagrams, Gauss sums, multiplicative characters, exponential sums, Kloosterman sums.

**AMS subject classifications.** 52C10

## 1. Introduction.

**1.1. The Erdős distance problem.** The Erdős distance conjecture in the Euclidean space says that if  $E$  is a finite subset of  $\mathbb{R}^d$ ,  $d \geq 2$ , then

$$(1.1) \quad \#\Delta(E) \gtrsim (\#E)^{\frac{2}{d}},$$

where

$$\Delta(E) = \{|x - y| : x, y \in E\},$$

with  $|x - y|^2 = (x_1 - y_1)^2 + \dots + (x_d - y_d)^2$  and here, and throughout the paper,  $X \lesssim Y$  means that there exists  $C > 0$  such that  $X \leq CY$ , and  $X \gtrsim Y$ , with the controlling parameter  $N$ , means that for every  $\epsilon > 0$  there exists  $C_\epsilon > 0$  such that  $X \leq C_\epsilon N^\epsilon Y$ .

Taking  $E = \mathbb{Z}^d \cap [0, N^{\frac{1}{d}}]^d$  shows that (1.1) cannot in general be improved. The conjecture has not been solved in any dimension. See, for example, [14], [1], and the references contained therein for the description of the conjecture, background material, and a survey of recent results.

In this paper we study the Erdős distance problem in vector spaces over finite fields. This problem was recently addressed by Tao ([19]) who relates it to some interesting questions in combinatorics, and, more recently, by Iosevich and Rudnev. We shall describe these results later in the introduction.

Let  $\mathbb{F}_q$  denote the finite field with  $q$  elements, and let  $\mathbb{F}_q^d$  denote the  $d$ -dimensional vector space over this field. Let  $E \subset \mathbb{F}_q^d$ ,  $d \geq 2$ . Then a possible analog of the classical Erdős distance problem is to determine the smallest possible cardinality of the set

$$\Delta_n(E) = \{||x - y||_n = (x_1 - y_1)^n + \dots + (x_d - y_d)^n : x, y \in E\},$$

with  $n$  a positive integer  $\geq 2$ , viewed as a subset of  $\mathbb{F}_q$ .

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In the finite field setting, the estimate (1.1) cannot hold without further restrictions. To see this, let  $E = \mathbb{F}_q^d$ . Then  $\#E = q^d$  and  $\#\Delta(E) = q$ . Furthermore, an interesting feature of the Erdős distance problem in the finite field setting with  $n = 2$  is the existence of non-trivial spheres of zero radius. These are sets of the form  $\{x \in \mathbb{F}_q^d : x_1^2 + x_2^2 + \dots + x_d^2 = 0\}$  and several assumptions in the statements of results below are there precisely to deal with issues created by the presence of this object. For example, suppose  $-1$  is a square in  $\mathbb{F}_q$ . Using spheres of zero radius one can show, in even dimensions, that there exists a set of cardinality precisely  $q^{\frac{d}{2}}$  such that all the distances,  $(x_1 - y_1)^2 + \dots + (x_d - y_d)^2$  are zero. What's more, suppose  $\mathbb{F}_q$  is a finite field, such that  $q = p^2$ , where  $p$  is a prime. Then  $E = \mathbb{F}_p^d$  is naturally embedded in  $\mathbb{F}_q^d$ , has cardinality  $q^{\frac{d}{2}}$ , and determines only  $\sqrt{q}$  distances. If  $n > 2$ , the situation is equally fascinating. For example, if  $n = 3$  and  $d = 2$ , the equation  $x_1^3 + x_2^3 = 0$  always has at least  $q$  solutions, since cube root of  $-1$  is  $-1$ . This equation may have as many as  $3q$  solutions if the primitive cube root of  $-1$  is in the field.

With these examples as guide, we generalize the conjecture originally stated in [10] in the case  $n = 2$  as follows.

**CONJECTURE 1.1.** *Let  $E \subset \mathbb{F}_q^d$  of cardinality  $\geq Cq^{\frac{d}{2}}$ , with  $C$  sufficiently large. Then*

$$\#\Delta_n(E) \gtrsim q.$$

The authors conjecture in [10] that the constant  $C$  that appears above may be taken to be any number bigger than one, at least in the case  $n = 2$ . It is interesting to note that if  $n > 2$ , the situation becomes more complicated. For example, as we pointed out above, if  $n = 3$  and  $d = 2$ , the number of points on the curve  $x_1^3 + x_2^3 = 0$  may be as high as  $3q$ , depending on whether or not the primitive cube root of  $-1$  is in the field. Thus a corresponding conjecture in the case  $n > 2$  must be designed with these issues in mind.

**2. Previous results.** A Euclidean plane argument due to Erdős ([6]) can be applied to the finite field set-up under the assumption of Conjecture 1.1 to show that if  $d = 2$  and  $\#E \geq Cq$ , with  $C$  sufficiently large, then

$$(2.1) \quad \#\Delta_n(E) \gtrsim (\#E)^{\frac{1}{2}}.$$

This result was improved by Bourgain, Katz and Tao ([3]) who showed using intricate incidence geometry that for every  $\epsilon > 0$ , there exists  $\delta > 0$ , such that if  $\#E \lesssim q^{2-\epsilon}$ , then

$$\#\Delta_2(E) \gtrsim q^{\frac{1}{2}+\delta}.$$

The relationship between  $\epsilon$  and  $\delta$  in the above argument is difficult to determine. Moreover, matters are even more subtle in higher dimensions in the context of vector spaces over finite fields, because intersection of analogs of spheres, both quadratic and cubic, in  $\mathbb{F}_q^d$  may be quite complicated, and the standard induction on the dimension argument in  $\mathbb{R}^d$  (see e.g. [1]) that allows one to bootstrap the estimate (2.1) into the estimate

$$(2.2) \quad \#\Delta_{\mathbb{R}^d}(E) \gtrsim (\#E)^{\frac{1}{d}}$$

does not immediately go through. We establish the finite field analog of the estimate (2.2) below using Fourier analytic methods and number theoretic properties of Kloosterman sums and its more general analogs.

Another way of thinking of Conjecture 1.1 is in terms of the Falconer distance conjecture, [7], in the Euclidean setting which says that if the Hausdorff dimension of a set in  $\mathbb{R}^d$  exceeds  $\frac{d}{2}$ , then the Lebesgue measure of the distance set is positive. Conjecture 1.1 implies that if the size of the set is greater than  $q^{\frac{d}{2}}$ , then the distance set contains a positive proportion of all the possible distances, an analogous statement.

In [10], the authors proved the following result.

**THEOREM 2.1.** *Let  $E \subset \mathbb{F}_q^d$ ,  $d \geq 2$ , such that  $\#E \geq Cq^{\frac{d+1}{2}}$ . Then if  $C$  is sufficiently large,  $\Delta_2(E)$  contains every element of  $\mathbb{F}_q$ .*

### 3. Main results of this paper.

**3.1. Distances determined by a single set.** Our first result is the version of Theorem 2.1 for cubic metrics.

**THEOREM 3.1.** *Suppose that  $q$  is a prime number congruent to 1 modulo 3. Let  $E \subset \mathbb{F}_q^d$ , such that  $\#E \geq Cq^{\frac{d+1}{2}}$ . Then if  $C$  is sufficiently large,  $\Delta_3(E)$  contains every element of  $\mathbb{F}_q$ .*

*Suppose that  $d = 2$ , and  $n \geq 2$ . Then if  $\#E \geq Cq^{\frac{3}{2}}$  for  $C$  sufficiently large, then  $\Delta_n(E)$  contains every elements of  $\mathbb{F}_q$ .*

**COROLLARY 3.2.** *Suppose that  $q$  is a prime number congruent to 1 modulo 3. Let  $E \subset \mathbb{F}_q^d$ ,  $d \geq 2$ , such that  $\#E = Cq^{\frac{d+1}{2}}$ . Then if  $C$  is sufficiently large,*

$$\#\Delta_3(E) \approx (\#E)^{\frac{2}{d+1}}.$$

*In two dimensions, the same conclusion, with  $d = 2$ , holds for any  $n \geq 2$ .*

Note that in the case  $d = 2$ , the exponent  $\frac{2}{3}$  obtained via the corollary, for the given range of parameters, is a much better exponent than the one obtained by the incidence argument due to Erdős described in (2.1) above. Also, we point out once more that Erdős' argument does not generalize to higher dimensions, at least not very easily, due to the possibly complicated intersection properties of cubic varieties.

**3.2. Szemerédi-Trotter type Incidence theorems and distances between pairs of sets.** As in the case  $n = 2$ , the proof of Theorem 3.1 can be modified to yield a good upper bound on the number of incidences between points and cubic surfaces in vector spaces over finite fields. It is an analog, and a higher dimensional generalization, of the following classical result due to Szemerédi and Trotter.

**THEOREM 3.3.** *The number of incidences between  $N$  points and  $M$  lines (or circles of the same radius) in the plane is*

$$\lesssim N + M + (NM)^{\frac{2}{3}}.$$

Our incident estimate is the following.

**THEOREM 3.4.** *Suppose that  $q$  is a prime number congruent to 1 modulo 3. Let  $E, F \subset \mathbb{F}_q^d$ ,  $d \geq 2$ . Then if  $j \neq 0$ ,*

$$\begin{aligned} & \#\{(x, y) \in E \times F : (x_1 - y_1)^3 + \dots + (x_d - y_d)^3 = j\} \\ & \lesssim \#E \cdot \#F \cdot q^{-1} + q^{\frac{d-1}{2}} \cdot (\#E)^{\frac{1}{2}} \cdot (\#F)^{\frac{1}{2}}. \end{aligned}$$

*Similarly, if  $q$  is a prime number and  $j \neq 0$ , then*

$$\begin{aligned} & \#\{(x, y) \in E \times F : (x_1 - y_1)^2 + \dots + (x_d - y_d)^2 = j\} \\ & \lesssim \#E \cdot \#F \cdot q^{-1} + q^{\frac{d-1}{2}} \cdot (\#E)^{\frac{1}{2}} \cdot (\#F)^{\frac{1}{2}}. \end{aligned}$$

*In two dimensions, the same result holds, with  $d = 2$ , with  $\Delta_3$  replaced by  $\Delta_n$  for any  $n \geq 2$ .*

**REMARK 3.5.** *In particular, if  $\#E \approx \#F \approx q^{\frac{d+1}{2}}$ , then the number of incidences between points in  $E$  and “spheres”, quadratic or cubic, centered at elements of  $F$  is  $\lesssim q^d$ .*

*To make the numerology more transparent, Theorem 3.4 says that if  $N \approx q^{\frac{d+1}{2}}$ , the number of incidences between  $\approx N$  points and  $\approx N$  spheres, cubic or quadratic, in  $\mathbb{F}_q^d$  is  $\lesssim q^d = N^{\frac{2d}{d+1}}$ . In two dimensions this says that the number of incidences between  $N$  points and  $N$  circles is  $\lesssim N^{\frac{4}{3}}$ , provided that  $N \approx q^{\frac{d+1}{2}}$ , matching in this setting the exponent in the celebrated result due to Szemerédi and Trotter in the Euclidean plane (see Theorem 3.3 above).*

An easy modification of the method used to prove Theorem 3.4 above yields the following distance set result.

**COROLLARY 3.6.** *Let  $E, F \subset \mathbb{F}_q^d$ ,  $d \geq 2$ . Suppose that  $q$  is a prime number congruent to 1 modulo 3 and  $\#E \cdot \#F \geq Cq^{d+1}$ . Let  $\Delta_3(E, F) = \{\|x - y\|_3 : x \in E, y \in F\}$ . Then if  $C$  is sufficiently large, then  $\Delta_3(E, F)$  contains every element of  $\mathbb{F}_q^*$ .*

*As before, in two dimensions the same conclusion holds, with  $d = 2$ , with  $\Delta_3$  replaced by  $\Delta_n(E)$ .*

Observe that if  $E = F$ , then we can safely say that in fact  $\Delta_3(E, F)$  contains every element of  $\mathbb{F}_q$ , but if  $E \neq F$ , the zero distance may not be present.

We also call the reader’s attention to the fact that an analogous version of this result was independently obtained by Shparlinski in [17].

**4. Fourier analytic preliminaries and notation.** Let  $\mathbb{F}_q$  be a finite field with  $q$  elements, where  $q$  is a prime number. Let

$$\chi(t) = e^{\frac{2\pi i}{q} t}.$$

Given a complex valued function  $f$  on  $\mathbb{F}_q^d$ , define the Fourier transform of  $f$  by the equation

$$\widehat{f}(m) = q^{-d} \sum_{x \in \mathbb{F}_q^d} \chi(-x \cdot m) f(x).$$

We also need the following basic identity, typically known as the Plancherel theorem. Let  $f$  be as above. Then

$$\sum_{m \in \mathbb{F}_q^d} |\widehat{f}(m)|^2 = q^{-d} \sum_{x \in \mathbb{F}_q^d} |f(x)|^2.$$

**5. Proof of the first part of Theorem 3.1.** Let  $\chi(s) = e^{\frac{2\pi i}{q}s}$ . Let  $S_j$  denote the characteristic function of the cubic sphere

$$\{x \in \mathbb{F}_q^d : \|x\|_3 = j\},$$

where, as above,

$$\|x\|_3 = x_1^3 + \dots + x_d^3.$$

The key estimate of the paper is the following.

**THEOREM 5.1.** *Let  $\|x\|_3 = x_1^3 + \dots + x_d^3$ . Suppose that  $q$  is a prime number congruent to 1 modulo 3 and  $j \neq 0$ . Then if  $m \neq (0, \dots, 0)$ , then*

$$\left| \widehat{S}_j(m) \right| = \left| q^{-d} \sum_{\{x \in \mathbb{F}_q^d : \|x\|_3 = j\}} \chi(x \cdot m) \right| \lesssim q^{-\frac{d+1}{2}},$$

and if  $m = (0, \dots, 0)$ , then

$$\widehat{S}_j(m) = q^{-1} + O(q^{-\frac{d+1}{2}}) \approx q^{-1}.$$

For  $j \neq 0$ , consider

$$\begin{aligned} & \#\{(x, y) \in E \times E : \|x - y\|_3 = j\} \\ &= \sum_{x, y \in \mathbb{F}_q^d} E(x)E(y)S_j(x - y) \\ &= q^{2d} \sum_m |\widehat{E}(m)|^2 \widehat{S}_j(m) = A + B, \end{aligned}$$

where

$$A = q^{2d} |\widehat{E}(0, \dots, 0)|^2 \widehat{S}_j(0, \dots, 0),$$

and

$$B = q^{2d} \sum_{m \neq (0, \dots, 0)} |\widehat{E}(m)|^2 \widehat{S}_j(m).$$

Using the second part of Theorem 5.1,

$$A \approx q^{2d} q^{-2d} (\#E)^2 \cdot q^{-1}.$$

Whereas using the first part of Theorem 5.1,

$$\begin{aligned} |\mathbf{B}| &\lesssim q^{2d} q^{-\frac{d+1}{2}} \sum_{m \neq (0, \dots, 0)} |\widehat{E}(m)|^2 \\ &\lesssim q^{2d} q^{-\frac{d+1}{2}} q^{-d} \sum_{x \in \mathbb{F}_q^d} E^2(x) = q^{\frac{d-1}{2}} \cdot \#E. \end{aligned}$$

We therefore obtain that

$$\#\{(x, y) \in E \times E : \|x - y\|_3 = j\} = \mathbf{A} + \mathbf{B},$$

where

$$\mathbf{A} \gtrsim (\#E)^2 q^{-1},$$

and

$$|\mathbf{B}| \lesssim \#E \cdot q^{\frac{d-1}{2}}.$$

We conclude that if  $\#E \geq Cq^{\frac{d+1}{2}}$ , with  $C$  sufficiently large, then

$$\#\{(x, y) \in E \times E : \|x - y\|_3 = j\} > 0$$

for each  $j \neq 0$ . This completes the proof of Theorem 3.1.

**6. Proof of Theorem 5.1.** We have

$$\begin{aligned} \widehat{S}_j(m) &= q^{-d} \sum_{\{x \in \mathbb{F}_q^d : \|x\|_3 = j\}} \chi(-x \cdot m) \\ &= q^{-1} \delta(m) + q^{-d-1} \sum_x \sum_{t \in \mathbb{F}_q^*} \chi(t(\|x\|_3 - j)) \chi(-x \cdot m), \end{aligned}$$

where  $\delta(m) = 1$  if  $m = (0, \dots, 0)$  and 0 otherwise.

LEMMA 6.1. *Let  $\chi$  be a nontrivial additive character of  $\mathbb{F}_q$  with  $q$  congruent to 1 modulo 3. Suppose that  $m = (m_1, \dots, m_l) \in (\mathbb{F}_q^*)^l$ . Then for any multiplicative character  $\psi$  of  $\mathbb{F}_q$  of order 3 and  $t \neq 0$ , we have*

$$\begin{aligned} &\prod_{j=1}^l \sum_{s_j \in \mathbb{F}_q} \chi(-s_j m_j + s_j^3 t) \\ &= \psi^{-l}(t) \sum_{s_1, \dots, s_l \in \mathbb{F}_q^*} \chi(s_1 + \dots + s_l + m_1^3 t^{-1} s_1^{-1} + \dots + m_l^3 t^{-1} s_l^{-1}) \psi(s_1) \cdots \psi(s_l), \end{aligned}$$

where  $3^{-3} m_j^3$  is denoted by  $m_j^3$  in the right-hand side of the equation.

We shall also need the following result due to Duke and Iwaniec ([4]).

THEOREM 6.2. *Suppose that  $q$  is congruent to 1 modulo 3 and let  $\psi$  be a multiplicative character of order three. Then*

$$\sum_{s \in \mathbb{F}_q} \chi(as^3 + s) = \sum_{s \in \mathbb{F}_q^*} \psi(sa^{-1})\chi(s - (3^3as)^{-1}),$$

for any  $a \in \mathbb{F}_q^*$ . It follows that

$$\begin{aligned} \sum_{s \in \mathbb{F}_q} \chi(-sm_j + s^3t) &= \sum_{s \in \mathbb{F}_q} \chi(s - s^3tm_j^{-3}) \\ &= \sum_{s \in \mathbb{F}_q^*} \psi(st^{-1})\chi(s + m_j^3t^{-1}3^{-3}s^{-1}). \end{aligned}$$

since  $\psi$  is a multiplicative character of  $\mathbb{F}_q$  of order three and  $m_j \neq 0$ . Absorbing  $3^{-3}$  into  $m_j$  to make the notations simple, we complete the proof of Lemma 6.1.

LEMMA 6.3. *Let  $\chi$  be a nontrivial additive character of  $\mathbb{F}_q$  with  $q$  congruent to 1 modulo 3. Then for any multiplicative character  $\psi$  of  $\mathbb{F}_q$  of order 3 and  $t \neq 0$ , we have*

$$\left( \sum_{s \in \mathbb{F}_q} \chi(ts^3) \right)^l = \sum_{r=0}^l \binom{l}{r} q^l \psi^{-(l+r)}(t) \left( \widehat{\psi}(-1) \right)^{l-r} \left( \widehat{\psi^2}(-1) \right)^r,$$

where  $\binom{\cdot}{\cdot}$  is a binomial coefficient,  $l$  is a positive integer, and the Fourier transform of a multiplicative character  $\psi$  of  $\mathbb{F}_q$  is given by

$$\widehat{\psi}(v) = q^{-1} \sum_{s \in \mathbb{F}_q^*} \chi(-vs)\psi(s).$$

REMARK 6.4.  $\widehat{\psi}(v) = O(q^{-\frac{1}{2}})$  for  $v \neq 0$ .

To prove Lemma 6.3, we need the following theorem. For the proof, see the ([13], page 217, Theorem 5.30).

THEOREM 6.5. *Let  $\chi$  be a nontrivial additive character of  $\mathbb{F}_q$ ,  $n \in \mathbb{N}$ , and  $\psi$  a multiplicative character of  $\mathbb{F}_q$  of order  $h = \gcd(n, q-1)$ . Then*

$$\sum_{s \in \mathbb{F}_q} \chi(ts^n + b) = \chi(b) \sum_{k=1}^{h-1} \psi^{-k}(t) G(\psi^k, \chi)$$

for any  $t, b \in \mathbb{F}_q$  with  $t \neq 0$ , where  $G(\psi^k, \chi) = \sum_{s \in \mathbb{F}_q^*} \psi^k(s)\chi(s)$ .

By using Theorem 6.5, we see that for any multiplicative character  $\psi$  of order three,

$$\left( \sum_{s \in \mathbb{F}_q} \chi(ts^3) \right)^l = \left( \sum_{k=1}^2 \psi^{-k}(t) \sum_{s \in \mathbb{F}_q^*} \psi^k(s)\chi(s) \right)^l$$

$$\begin{aligned}
&= \left( \psi^{-1}(t) \sum_{s \in \mathbb{F}_q^*} \psi(s) \chi(s) + \psi^{-2}(t) \sum_{s \in \mathbb{F}_q^*} \psi^2(s) \chi(s) \right)^l \\
&= \left( G_1(t) + G_2(t) \right)^l = \sum_{r=0}^l \binom{l}{r} G_1(t)^{l-r} G_2(t)^r,
\end{aligned}$$

where

$$G_1(t) = \psi^{-1}(t) \sum_{s \in \mathbb{F}_q^*} \psi(s) \chi(s)$$

and

$$G_2(t) = \psi^{-2}(t) \sum_{s \in \mathbb{F}_q^*} \psi^2(s) \chi(s).$$

Note that  $G_1(t) = q\psi^{-1}(t)\widehat{\psi}(-1)$  and  $G_2(t) = q\psi^{-2}(t)\widehat{\psi^2}(-1)$ . Thus we conclude that

$$\left( \sum_{s \in \mathbb{F}_q} \chi(ts^3) \right)^l = \sum_{r=0}^l \binom{l}{r} q^l \psi^{-(l+r)}(t) \left( \widehat{\psi}(-1) \right)^{l-r} \left( \widehat{\psi^2}(-1) \right)^r.$$

We are now ready to prove Theorem 5.1. First, we assume that  $m = (0, \dots, 0) \in \mathbb{F}_q^d$ . Then, using Lemma 6.3, we see that

$$\begin{aligned}
\widehat{S}_j(0, \dots, 0) &= q^{-d} \sum_{\{x \in \mathbb{F}_q^d : \|x\|_3 = j\}} 1 \\
&= q^{-1} + q^{-d-1} \sum_{t \in \mathbb{F}_q^*} \chi(-tj) \sum_x \chi(t(\|x\|_3)) \\
&= q^{-1} + q^{-d-1} \sum_{t \in \mathbb{F}_q^*} \chi(-tj) \sum_{r=0}^d \binom{d}{r} q^d \psi^{-(d+r)}(t) \left( \widehat{\psi}(-1) \right)^{d-r} \left( \widehat{\psi^2}(-1) \right)^r \\
&= q^{-1} + q^{-1} \sum_{r=0}^d \binom{d}{r} \left( \widehat{\psi}(-1) \right)^{d-r} \left( \widehat{\psi^2}(-1) \right)^r \sum_{t \in \mathbb{F}_q^*} \chi(-tj) \psi^{-(d+r)}(t) \\
&= q^{-1} + q^{-1} \sum_{r=0}^d \binom{d}{r} \left( \widehat{\psi}(-1) \right)^{d-r} \left( \widehat{\psi^2}(-1) \right)^r q \widehat{\psi^{-(d+r)}}(j) \\
&= q^{-1} + O(q^{-\frac{d+1}{2}}) \approx q^{-1}.
\end{aligned}$$



In the last equality, we used the fact that  $\widehat{\psi}(v) = O(q^{-\frac{1}{2}})$  for any multiplicative character of  $\mathbb{F}_q$  with  $v \neq 0$ . Thus the second part of Theorem 5.1 is proved.

In order to prove the first part of Theorem 5.1, we shall deal with the problem in case  $m = (m_1, \dots, m_d) \neq (0, \dots, 0)$ . Suppose that  $m_j \neq 0$  for  $j \in J \subset \{1, 2, \dots, d\}$  and  $m_j = 0$  for  $j \in \{1, 2, \dots, d\} \setminus J = J'$ . Without loss of generality, we may assume that  $J = \{1, 2, \dots, l\}$  and  $J' = \{l+1, \dots, d\}$  for some  $l = 1, 2, \dots, d$ . Using Lemma 6.1 and Lemma 6.3, we see that

$$\begin{aligned}
\widehat{S}_j(m) &= q^{-d-1} \sum_{t \in \mathbb{F}_q^*} \chi(-tj) \sum_{x \in \mathbb{F}_q^d} \chi(t\|x\|_3 - m \cdot x) \\
&= q^{-d-1} \sum_{t \in \mathbb{F}_q^*} \chi(-tj) \left( \prod_{k=1}^l \sum_{s_k \in \mathbb{F}_q} \chi(ts_k^3 - m_k s_k) \right) \left( \prod_{k=l+1}^d \sum_{s_k \in \mathbb{F}_q} \chi(ts_k^3) \right) \\
&= q^{-d-1} \sum_{t \in \mathbb{F}_q^*} \chi(-tj) \psi^{-l}(t) \sum_{s_1, \dots, s_l \in \mathbb{F}_q^*} \chi(s_1 + \dots + s_l + m_1^3 t^{-1} s_1^{-1} + \dots + m_l^3 t^{-1} s_l^{-1}) \psi(s_1) \cdots \psi(s_l) \\
&\quad \times \sum_{r=0}^{d-l} \binom{d-l}{r} q^{d-l} \psi^{-(d-l+r)}(t) \left( \widehat{\psi}(-1) \right)^{d-l-r} \left( \widehat{\psi^2}(-1) \right)^r \\
&= q^{-1-l} \sum_{r=0}^{d-l} \binom{d-l}{r} \left( \widehat{\psi}(-1) \right)^{d-l-r} \left( \widehat{\psi^2}(-1) \right)^r \sum_{t \in \mathbb{F}_q^*} \chi(-tj) \psi^{-(d+r)}(t) \\
&\quad \times \sum_{s_1, \dots, s_l \in \mathbb{F}_q^*} \chi(s_1 + \dots + s_l + m_1^3 t^{-1} s_1^{-1} + \dots + m_l^3 t^{-1} s_l^{-1}) \psi(s_1) \cdots \psi(s_l).
\end{aligned}$$

Since  $\binom{d-l}{r} \left( \widehat{\psi}(-1) \right)^{d-l-r} \left( \widehat{\psi^2}(-1) \right)^r = O(q^{-\frac{1}{2}(d-l)})$ , we obtain that

$$\left| \widehat{S}_j(m) \right| \lesssim q^{-1-\frac{d+l}{2}} \sum_{r=0}^{d-l} |A_r(\chi, \psi)|,$$

where  $A_r(\chi, \psi)$  is given by

$$\sum_{t \in \mathbb{F}_q^*} \chi(-tj) \psi^{-(d+r)}(t) \sum_{s_1, \dots, s_l \in \mathbb{F}_q^*} \chi(s_1 + \dots + s_l + m_1^3 t^{-1} s_1^{-1} + \dots + m_l^3 t^{-1} s_l^{-1}) \psi(s_1) \cdots \psi(s_l).$$

We now apply the result of Adolphson and Sperber ([2], Theorem 4.2, Corollary 4.3) to see that for all  $r = 0, 1, \dots, d-l$ ,

$$|A_r(\chi, \psi)| \lesssim q^{\frac{l+1}{2}}.$$

This completes the proof.

**7. Proof of the second part of Theorem 3.1.** As in the proof of the first part of Theorem 3.1, it suffices to prove the following estimation.

**THEOREM 7.1.** *Let  $\|x\|_n = x_1^n + x_2^n$  for  $x \in \mathbb{F}_q^2$  and  $n \geq 2$ . Suppose that  $q$  is a prime number and  $j \neq 0$ . Then if  $m \neq (0, 0)$ , then*

$$\left| \widehat{S}_j(m) \right| = \left| q^{-2} \sum_{\{x \in \mathbb{F}_q^2: \|x\|_n=j\}} \chi(-x \cdot m) \right| \lesssim q^{-\frac{3}{2}},$$

and if  $m = (0, 0)$ , then

$$\widehat{S}_j(m) = q^{-1} + O(q^{-\frac{3}{2}}) \approx q^{-1}.$$

To prove Theorem 7.1, we observe that for  $j \neq 0$  and  $m \in \mathbb{F}_q^2$ ,

$$\begin{aligned} \widehat{S}_j(m) &= q^{-2} \sum_{\{x \in \mathbb{F}_q^2: \|x\|_n=j\}} \chi(-x \cdot m) \\ &= q^{-1} \delta(m) + q^{-3} \sum_x \sum_{t \in \mathbb{F}_q^*} \chi(t(\|x\|_n - j)) \chi(-x \cdot m), \end{aligned}$$

where  $\delta(m) = 1$  if  $m = (0, 0)$  and 0 otherwise.

First we shall prove the second part of Theorem 7.1. Let  $\psi$  be a multiplicative character of  $\mathbb{F}_q$  of order  $h = \gcd(n, q-1)$ . For each  $i = 1, 2, \dots, (h-1)$ , we denote by  $\beta_i$  a non-negative integer. Then by Theorem 6.5, we see that

$$\begin{aligned} &\left( \sum_{s \in \mathbb{F}_q} \chi(ts^n) \right)^2 \\ &= \sum_{\beta_1 + \dots + \beta_{h-1} = 2} \frac{2!}{\beta_1! \dots \beta_{h-1}!} \psi^{-(\beta_1 + \dots + (h-1)\beta_{h-1})}(t) q^2 \left( \widehat{\psi}(-1) \right)^{\beta_1} \dots \left( \widehat{\psi^{h-1}}(-1) \right)^{\beta_{h-1}}. \end{aligned}$$

It therefore follows that

$$\widehat{S}_j(0, 0) = q^{-1} + \sum_{\beta_1 + \dots + \beta_{h-1} = 2} \frac{2!}{\beta_1! \dots \beta_{h-1}!} \psi^{-\gamma(h, \beta)}(j) \left( \widehat{\psi}(-1) \right)^{\beta_1} \dots \left( \widehat{\psi^{h-1}}(-1) \right)^{\beta_{h-1}}$$

where  $\gamma(h, \beta)$  is given by  $\beta_1 + 2\beta_2 + \dots + (h-1)\beta_{h-1}$ .

Since  $\widehat{\psi}(v) = O(q^{-\frac{1}{2}})$  for each multiplicative character  $\psi$  and  $v \in \mathbb{F}_q^*$ , we conclude

$$\widehat{S}(0, 0) = q^{-1} + O(q^{-\frac{3}{2}}) \approx q^{-1}.$$

This completes the proof of the second part of Theorem 7.1.

It remains to prove the first part of Theorem 7.1. The cohomological interpretation can be used to estimate the exponential sums. We now introduce the cohomology theory based on work of authors in [5] and [2]. Let  $g$  be a polynomial given by

$$(7.1) \quad g = \sum_{\alpha \in J} A_\alpha x^\alpha \in \mathbb{F}_q[x_1, \dots, x_d],$$

where  $J$  is a finite subset of  $(\mathbb{N} \cup \{0\})^d$ , and  $A_\alpha \neq 0$  if  $\alpha \in J$ . We denote by  $\sum(g)$  the Newton polyhedron of  $g$  which is the convex hull in  $\mathbb{R}^d$  of the set  $J \cup (0, \dots, 0)$ . For any face  $\sigma$  (of any dimension) of  $\sum(g)$ , we put

$$g_\sigma = \sum_{\alpha \in \sigma \cap J} A_\alpha x^\alpha.$$

DEFINITION 7.2. *Let  $g \in \mathbb{F}_q[x_1, \dots, x_d]$  be a polynomial as in (7.1). We say that  $g$  is nondegenerate with respect to  $\sum(g)$  if for every face  $\sigma$  of  $\sum(g)$  that does not contain the origin, the polynomials*

$$\frac{\partial g_\sigma}{\partial x_1}, \dots, \frac{\partial g_\sigma}{\partial x_d}$$

have no common zero in  $(\overline{\mathbb{F}_q}^*)^d$  where  $\overline{\mathbb{F}_q}$  denotes an algebraic closure of  $\mathbb{F}_q$ . We say that  $g$  is commode with respect to  $\sum(g)$  if for each  $k = 1, 2, \dots, d$ ,  $g$  contains a term  $A_k x_k^{\alpha_k}$  for some  $\alpha_k > 0$  and  $A_k \neq 0$ .

The general version of the following theorem can be found in [5] (see Theorem 9.2).

THEOREM 7.3. *Let  $q$  be a prime number. Suppose that  $g : \mathbb{F}_q^d \rightarrow \mathbb{F}_q$ ,  $d \geq 2$ , is commode and nondegenerate with respect to  $\sum(g)$ . Then*

$$\sum_{x \in \mathbb{F}_q^d} \chi(g(x)) = O(q^{\frac{d}{2}}).$$

We now prove the first part of Theorem 7.1. Since  $m \neq (0, 0)$ , we have  $\sum_{x \in \mathbb{F}_q^2} \chi(-x \cdot m) = 0$ . We therefore see that for  $j \neq 0$ ,

$$\widehat{S}_j(m) = q^{-3} \sum_{(t, x_1, x_2) \in \mathbb{F}_q^* \times \mathbb{F}_q^2} \chi(g(t, x_1, x_2)) = q^{-3} \sum_{(t, x_1, x_2) \in \mathbb{F}_q^3} \chi(g(t, x_1, x_2)),$$

where  $g(t, x_1, x_2) = tx_1^n + tx_2^n - m_1x_1 - m_2x_2 - jt$ .

If  $m_1 \cdot m_2 \neq 0$ , then  $g$  is commode. By Theorem 7.3, it suffices to show that  $g$  is nondegenerate with respect to  $\sum(g)$ . Note that  $\sum(g)$  has five zero-dimensional faces, eight one-dimensional faces and three two-dimensional faces which do not contain the origin. It is easy to show that for every face  $\sigma$  of  $\sum(g)$  that does not contain the origin, the polynomials

$$\frac{\partial g_\sigma}{\partial t}, \frac{\partial g_\sigma}{\partial x_1}, \frac{\partial g_\sigma}{\partial x_2}$$

have no common zero in  $(\mathbb{F}_q^*)^3$  because we may assume that  $q$  is sufficiently large and so  $n$  is not congruent to 0 modulo  $q$ . This implies that  $g$  is nondegenerate with respect to  $\sum(g)$ . We now assume that  $m_1 \cdot m_2 = 0$ . Without loss of generality, we may assume that  $m_1 \neq 0$ , and  $m_2 = 0$  because  $m \neq (0, 0)$ . By using Theorem 6.5, we obtain that for a multiplicative character  $\psi$  of  $\mathbb{F}_q$  of order  $h = \gcd(n, q-1)$ ,

$$\widehat{S}_j(m) = q^{-3} \sum_{(t, x_1) \in \mathbb{F}_q^* \times \mathbb{F}_q} \chi(tx_1^n - m_1x_1 - jt) \sum_{k=1}^{h-1} \psi^{-k}(t) q \widehat{\psi}^k(-1)$$

$$\begin{aligned}
&= q^{-2} \sum_{k=1}^{h-1} \widehat{\psi^k}(-1) \sum_{(t, x_1) \in \mathbb{F}_q^* \times \mathbb{F}_q} \psi^{-k}(t) \chi(tx_1^n - m_1 x_1 - jt) \\
&\lesssim q^{-2} q^{-\frac{1}{2}} \sum_{k=1}^{h-1} |R_k(\psi^{-k}, \chi)|,
\end{aligned}$$

where  $R_k(\psi^{-k}, \chi)$  is given by

$$\sum_{(t, x_1) \in \mathbb{F}_q^* \times \mathbb{F}_q} \psi^{-k}(t) \chi(tx_1^n - m_1 x_1 - jt).$$

For each  $k = 1, 2, \dots, h-1$ , define  $\psi^{-k}(0) = 0$ . Then we can obtain that

$$R_k(\psi^{-k}, \chi) = \sum_{(t, x_1) \in \mathbb{F}_q^* \times \mathbb{F}_q} \psi^{-k}(t) \chi(tx_1^n - m_1 x_1 - jt).$$

Applying Theorem 7.3, we have

$$R_k(\psi^{-k}, \chi) = O(q).$$

This completes the proof.

**8. Proof of Theorem 3.4 and Corollary 3.6.** As we mentioned in the introduction, this is a simple variation on the proof of Theorem 3.1. Indeed,

$$\begin{aligned}
&\#\{(x, y) \in E \times F : \|x - y\|_n = j\} \\
&= q^{2d} \sum_m \overline{\widehat{E}(m)} \widehat{F}(m) \widehat{S}_j(m) \\
&= \#E \cdot \#F \cdot \widehat{S}_j(0, \dots, 0) + q^{2d} \sum_{m \neq (0, \dots, 0)} \overline{\widehat{E}(m)} \widehat{F}(m) \widehat{S}_j(m) = \text{I} + \text{II}.
\end{aligned}$$

By the second part of Theorem 5.1 (or Theorem 7.1),

$$\text{I} \lesssim \#E \cdot \#F \cdot q^{-1}.$$

Applying Cauchy-Schwartz, Theorem 5.1 (or Theorem 7.1) and Plancherel, we see that

$$\begin{aligned}
|\text{II}| &\lesssim q^{2d} q^{-\frac{d+1}{2}} \sum_{m \neq (0, \dots, 0)} |\widehat{E}(m)| |\widehat{F}(m)| \\
&\leq q^{2d} q^{-\frac{d+1}{2}} \left( \sum_m |\widehat{E}(m)|^2 \right)^{\frac{1}{2}} \cdot \left( \sum_m |\widehat{F}(m)|^2 \right)^{\frac{1}{2}} \\
&\leq q^{2d} q^{-\frac{d+1}{2}} q^{-d} \left( \sum_x |E(x)|^2 \right)^{\frac{1}{2}} \cdot \left( \sum_x |F(x)|^2 \right)^{\frac{1}{2}}
\end{aligned}$$

$$= q^{\frac{d-1}{2}} \cdot \sqrt{\#E} \cdot \sqrt{\#F}.$$

This completes the proof of Theorem 3.4.

In order to prove Corollary 3.6, we observe that by the second part of Theorem 5.1 (or Theorem 7.1),

$$I \gtrsim \#E \cdot \#F \cdot q^{-1}.$$

On the other hand, we have seen above that

$$|II| \lesssim q^{\frac{d-1}{2}} \cdot \sqrt{\#E} \cdot \sqrt{\#F},$$

and the result follows by a direct comparison.

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